ELECTRIC WAVE FILTERS HAVING TERMINATED FILTERS AS ARMS

By

Manley C. Osborne

Thesis
O82

Library
U. S. Naval Postgraduate School
Annapolis, Md.

ELECTRIC WAVE FILTERS HAVING
TERMINATED FILTERS
AS ARMS

By

Manley C. Osborne

SUBMITTED IN PARTIAL FULLFILLMENT OF THE
REQUIREMENT FOR THE DEGREE OF
MASTER OF SCIENCE IN ELECTRICAL ENGINEERING
AT THE
MOORE SCHOOL OF ELECTRICAL ENGINEERING
UNIVERSITY OF PENNSYLVANIA

PHILADELPHIA, PENNSYLVANIA MAY 14,1949

ACKNOWLEDGMENT

The writer is greatly indebted to Dr. J.G. Brainerd for suggestion of the topic and for helpful criticism during the development.

12623

TABLE OF CONTENTS

Page

I Introduction 1

II Conclusions 1

III Theory 3

  Lattice arrangement 4

    Types of arms 4

    Example with Low-Pass Arms, constant-K sections 6

    Example with High-Pass Arms, constant-K sections 7

    Examples with m-derived, constant-K sections 8

    Effect of different cutoff frequencies of arms 11

  Tee arrangement 15

    Examples with Low-Pass Arms 15,18

    Examples with High-Pass Arms 17,19

    Examples with m-derived, constant-K sections 20

    Method of increasing attenuation 23

    Effect of different cutoff frequencies of arms 25

  Miscellaneous cases 28

IV PLATES

  I Attenuation for lattice types 32

  II Lattice filter, effect of different cutoff frequencies 33

  III Tee filter, effect of different cutoff frequencies 34

V APPENDIX

  A. Types of arms 35

  B. Symbols 37

  C. Proof 38

  D. Proof 39

VI BIBLIOGRAPHY 40

I. INTRODUCTION

The object of this thesis is the study of the filter formed by using terminated filters (considered as two-terminal networks), rather than the usual pure reactances, as arms. It was hoped that such a filter would exhibit unusual properties which might prove useful, such as greater attenuation through a large band, steeper attenuation curves at cutoff, better impedance characteristics, absorption of power rather than reflection in the rejection bands, and economy of components.

Filters in parallel and filters in series have been previously analyzed[*2,*6], but a study of an arrangement like this has not been made previously, as far as can be determined. However, previous filter theory (a) is used to derive the results.

II. CONCLUSIONS

The properties of the derived filters are most interesting. The properties of many do not appear to be as good as in the usual arrangement, but in a number of cases described there are definite advantages to this new type filter. The most promising type is the lattice arrangement of two-terminal terminated filters. This type can be made to have a high attenuation over a large part of the rejection band (at least 43.5 db. over 97% of the band using MM' terminated arms). The most important property, however, is that this filter has a constant

---

[*2,*6] refers to numbers in Bibliography, page 40.

(a) McILWAIN and BRAINERD (Reference 3) is an excellent summary for the tee or pi section type. The reactance curves are most useful. GUILLEMIN (Reference 2) well covers the lattice type.

iterative impedance which is a pure resistance to all frequencies, pass or rejection. This means that this filter can be perfectly terminated, that there will be no reflections to the generator for either the pass or rejection frequencies (which always occur with the usual filter), and that the filter itself will absorb the rejection band power.

The tee arrangement leads to more complicated results than the lattice arrangement. Rejection band power is absorbed for only part of the rejection band. The iterative impedance of the section depends on the iterative impedance of its arms and is not an easy function to match. Whereas for the lattice type, low-pass arms always result in a high-pass filter and vice-versa, such is not true for the tee type. As an example of the tee type, low-pass series arms of pi termination and a low-pass shunt arm of the tee termination result in a high-pass filter. However, low-pass series arms of the tee termination and a low-pass shunt arm of the pi termination result in a band-pass filter.

Hence, we have the unusual result that given two low-pass arms of tee termination and two low-pass arms of pi-termination, we can construct

(a) a high-pass filter (using lattice arrangement)

(b) a high-pass filter with different cutoff frequency than previous filter (using tee arrangement with pi terminations in the series arms)

(c) a band-pass filter (using tee arrangement with tee terminations in the series arms)

(d) a low-pass filter (using the arms in conventional manner)

by switching alone.

Even though this new filter uses many components (22 capacitors and 22 inductors for a lattice with MM' arms, or 10 capacitors and 10 inductors using m-derived arms) it may be considered to have economy of components in some ways because of (1) the constant input impedance of the lattice type and (2) the flexibility of types that can be constructed with the same arms.

In all the above cases it has been assumed that the cutoff frequencies for all arms is the same. For both the lattice and tee arrangements an investigation has been made of the effect of different cutoff frequencies for the shunt and series arms. It is shown for the lattice type that the attenuation is low for the interval between the cutoff frequencies of the arms. For the tee type the attenuation is not affected adversely, other than shift of cutoff frequency.

The attenuation characteristic rather than the phase characteristic has been emphasized throughout as the attenuation characteristics determine the phase characteristics[*1].

The theory of this paper can be extended further; i.e., by using filters herein discussed (but not the lattice type) as arms of other filters, but this has not been investigated.

THEORY

The use of two-terminal terminated filters as arms of the filter results in arms which are apparently resistances to some frequencies and therefore (since they are terminated in a resistance) absorb that power, and apparently reactances to other frequencies. By making the shunt and series arms inverse networks, the reactance of the shunt arm will always be of opposite sign to the reactance of the series arm. If these arms are placed in a lattice arrangement we will have an all-pass network for those frequencies at which the arms appear to be reactances[a].

To simplify the discussion, all reactances shall be assumed pure. Further, it shall be assumed that the input impedance of each arm (each arm being a two-terminal filter) is equal to the iterative impedance of that type arm. This can be achieved theoretically by making the arms have an infinite number of sections, or more practically by assuming each arm to consist of only one section properly terminated at its far end, using a terminating section before the terminating resistor if necessary. With realizable terminating sections this will lead to only small errors over most of the pass-band of the arms. Appendix A illustrates all types of arms that will be discussed.

*3,p320

The iterative impedance of a tee section is

(1) $Z_T = K\sqrt{1+\rho_o}$

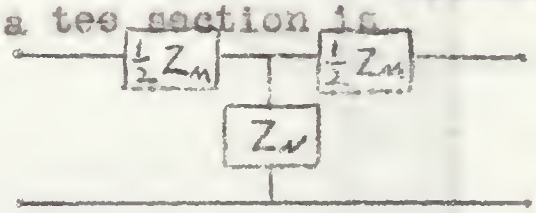


Fig.1

where K is the square root of the product of the total series and shunt reactances of the section, and where ρ_0 is $\frac{Z_m}{4Z_v}$, the total series impedance divided by four times the total shunt impedanwe[(a)].

---

[a] See Appendix C for proof that attenuation is zero when shunt and series arms of lattice are opposite reactances.

The iterative impedance of a pi section is[*3]

$$(2)\quad Z_\pi = \frac{K}{\sqrt{1+\rho_0}}$$

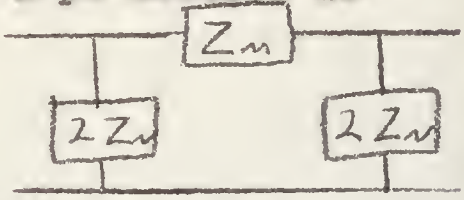


Fig.2

since we assumed inverse networks (and constant-K sections).

Let us now consider a lattice section:

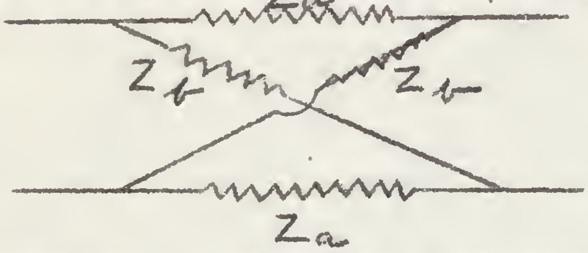


Fig.3

The attenuation of a lattice section is

$$(3)\quad \tanh\left(\frac{\alpha}{2}\right) = \sqrt{\frac{Z_b}{Z_a}} \qquad \text{if} \quad Z_b < Z_a$$

or

$$(4)\quad \tanh\left(\frac{\alpha}{2}\right) = \sqrt{\frac{Z_a}{Z_b}} \qquad \text{if} \quad Z_a < Z_b$$

where Z_a and Z_b are like impedances (both apparent resistances in this case)[*6,p239].

Let us now place the described pi terminated filters as the series arms of the lattice and the described tee terminated filters as the shunt arms of the lattice. The attenuation α of the lattice is then, by (3),

$$(5)\quad \tanh\left(\frac{\alpha}{2}\right) = \sqrt{\frac{K\sqrt{1+\rho_0}}{\frac{K}{\sqrt{1+\rho_0}}}} = \sqrt{1+\rho_0}$$

(a) See Appendix B for meaning of all symbols.

providing $\sqrt{1+\rho_0} \leq 1$. If $\sqrt{1+\rho_0} > 1$,

(6) $\tanh\left(\frac{\alpha}{2}\right) = \frac{1}{\sqrt{1+\rho_0}}$

#6,p238

The iterative impedance of any lattice is

(7) $Z_L = \sqrt{Z_a Z_b}$

Therefore the iterative impedance of our lattice is

(8) $Z_L = \sqrt{\frac{K}{\sqrt{1+\rho_0}} \cdot K\sqrt{1+\rho_0}} = K$

Thus we see the iterative impedance is a constant independent of frequency.

As special cases of the lattice type, let us consider the tee and pi arms to be constant-K, low-pass. Then

(9) $\rho_0 = \frac{j\omega L}{4\frac{1}{j\omega C}} = -\frac{1}{4}\omega^2 LC$

Therefore

(10) $\tanh\left(\frac{\alpha}{2}\right) = \sqrt{1-\frac{1}{4}\omega^2 LC}$

this equation being true, of course, only when the quantity under the radical is positive.

Figures 4 and 5 show the attenuation and iterative impedance vs. frequency for this type filter. The attenuation curve is also shown in Plate 1. The attenuation curve was computed using (10).

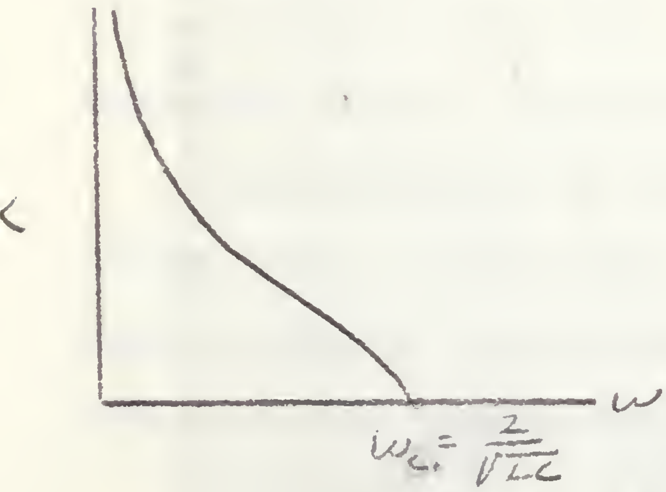


Fig. 4

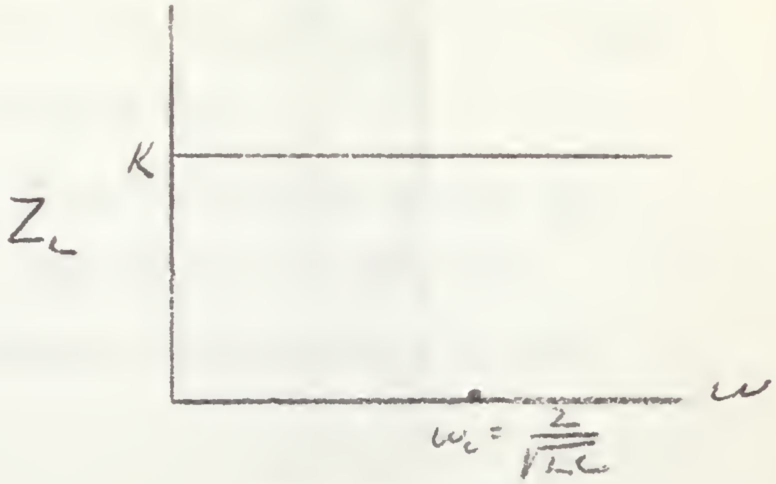


Fig. 5

As another special case of the lattice type, let us consider the tee and pi arms to be constant-K, high-pass. Then

(11) $\rho_0 = \frac{\frac{1}{j\omega C}}{4j\omega L} = -\frac{1}{4\omega^2 LC}$

Therefore

(12) $\tanh\left(\frac{\alpha}{2}\right) = \sqrt{1 - \frac{1}{4\omega^2 LC}}$

this equation being true only when the quantity under the radical is positive.

Figures 6 and 7 show the attenuation and iterative impedance for this type filter.

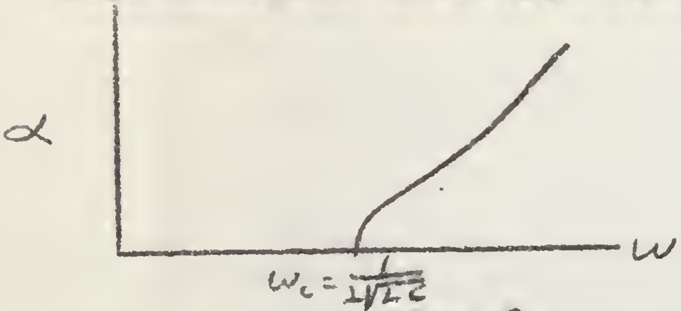


Fig. 6

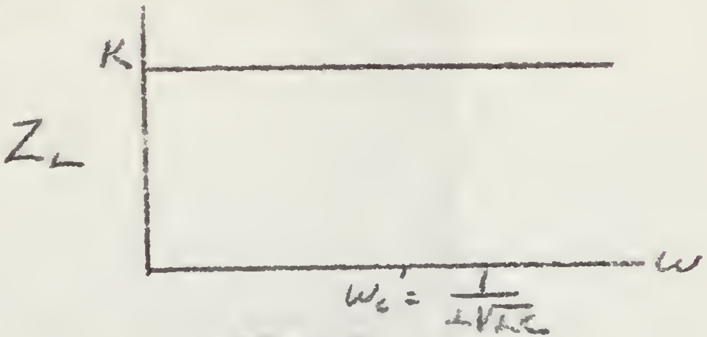


Fig. 7

It can be seen that for the lattice, using constant-K arms, shunt and series arms being inverse, attenuation will always exist when the arm input impedance appears to be a resistance (pass band for the arms), and only then, no matter how complicated the arms. Thus band-pass arms result in a band-rejection filter, etc. for this lattice type.

The attenuation of the two previous filters, using tee and pi arms, was low near cutoff, and in fact rather low through most of the attenuation range, but had one pole of infinite attenuation. Since

(3) $\tanh\left(\frac{\alpha}{2}\right) = \sqrt{\frac{Z_b}{Z_a}}$ and

(13) $\tanh^{-1} 1 = \infty$

to make a better attenuation characteristic we need to make $\frac{Z_i}{Z_a}$ be as near 1 as possible throughout the pass band of the arms. Arrangements of the m-derived type filters will accomplish this purpose better than the constant-K prototype. Curves of the mid-shunt iterative impedance of a mid-series m-derived type filter are given in a number of references #2,#3,#4,#6 and this same curve can be used to illustrate the mid-series iterative impedance of the mid-shunt m-derived type section #3. Further, the product of the mid-shunt iterative impedance of the mid-series m-derived type section by the mid-series iterative impedance of the mid-shunt m-derived section equals K^2.(See#3 or 4) These two networks are therefore inverse and to those frequencies at which the iterative impedances are reactances, the reactances of the two will be opposite in sign, as desired.

The mid-shunt iterative impedance of the series-derived m-type section is #4,p247

(14) $Z_{\pi}' = \frac{K}{\sqrt{1+\rho_0}}\left[1+(1-m^2)\rho_0\right]$

The mid-series iterative impedance of the shunt-derived m-type section is #4,p251

(15) $Z_T' = \frac{K\sqrt{1+\rho_0}}{1+(1-m^2)\rho_0}$

If we place the terminated filters as series arms of a lattice and the terminated filters as shunt arms of the lattice,

(16) $Z_L = \sqrt{Z_{\pi}' Z_T'} = \sqrt{\frac{K}{\sqrt{1+\rho_0}}\left[1+(1-m^2)\rho_0\right]\frac{K\sqrt{1+\rho_0}}{\left[1+(1-m^2)\rho_0\right]}} = K$

The attenuation α is

(17) $$\tanh\left(\frac{\alpha}{2}\right) = \sqrt{\frac{\dfrac{K\sqrt{1+\rho_0}}{1+(1-m^2)\rho_0}}{\dfrac{K}{\sqrt{1+\rho_0}}\left[1+(1-m^2)\rho_0\right]}} = \frac{\sqrt{1+\rho_0}}{1+(1-m^2)\rho_0}$$

We see that a new pole of infinite attenuation is introduced when

(18) $$\frac{\sqrt{1+\rho_0}}{1+(1-m^2)\rho_0} = 1$$

and this second pole can be moved by choosing m.

Plate 1 shows attenuation vs. ρ_0 for several values of m as a parameter. These curves could have been calculated using (17) but were actually calculated, for convenience, using curves of mid-shunt iterative impedance of the mid-series m-derived type vs. ρ_0 as given in Terman #6,p233. This can be done because this curve gives $\frac{Z_T'}{K}$ or $\frac{K}{Z_\pi'}$ vs. ρ_0, and

(19) $$\tanh\left(\frac{\alpha}{2}\right) = \sqrt{\frac{Z_T'}{Z_\pi'}} = \sqrt{\frac{\text{ORDINATE}\cdot K}{\dfrac{K}{\text{ORDINATE}}}} = \text{ordinate if ordinate} < 1$$

or (20) $$\tanh\left(\frac{\alpha}{2}\right) = \frac{1}{\text{ORDINATE}} \quad \text{if ordinate} > 1$$

Using a small value of m we can obtain an attenuation pole as near as desired to the cutoff frequency, but when m becomes very small the pole becomes very narrow at its base and the overall attenuation suffers. Values between $m \doteq .3$ and $m \doteq .6$ seem very good. In the usual filter theory the value of m is usually decided, at least in terminating sections, to give the optimum iterative impedance. In this lattice arrangement the iterative impedance is

(21) $$Z_L = \sqrt{Z_T' Z_\pi'} = \sqrt{\frac{K\sqrt{1+\rho_0}}{1+(1-m^2)\rho_0}\cdot\frac{K\left[1+(1-m^2)\rho_0\right]}{\sqrt{1+\rho_0}}} = K$$

a constant for all frequencies not affected by the value of m.

The attenuation of the lattice type filter can be improved further by the use of more complicated arms. Since "Theoretically, with dissipation neglected, the solution of the terminal wave-filter impedance problem---can be carried to any degree of proximation desired toward a constant resistance terminal image impedance in all transmitting bands"#7 with one lattice section we can approach infinite attenuation over the entire rejection band (Pass band of arms) to any degree of approximation. The MM' type of arms seems to be the most complicated type that is economically realizable, though Zobel #7 describes MM'M'' and more complicated types. In all of these more complicated types it is still possible to realize inverse networks and in Shea #4,p274 it is proved the mid-series image impedance characteristic of the shunt-derived MM' structure is inverse to the mid-shunt impedance of the series-derived MM' structure. Since the MM' termination can be designed to give an iterative impedance constant to within 2% over 97% of the pass band. #2,p351 using the reasoning of (19), the minimum attenuation over 97% of the rejection band of the lattice filter is

$$\tanh\left(\frac{\alpha}{2}\right) = 1 - .02 \quad \text{or } \alpha = 5.11 \text{ napiers} = 43.5 \text{ db.}$$

Plate 1 shows the attenuation characteristic of a lattice with MM' terminated filters as arms, with M=.7230 and M'=.4134. There are three poles of infinite attenuation. This curve could have been computed in the manner of previous curves, but for convenience the reasoning of (19) was applied to the iterative impedance characteristics curve of the MM' type as drawn by

Zobel on page 314 of #7.

Examination of Plate 1 shows that a lattice filter with MM' arms is little or no more efficient than two lattice sections with m-derived arms (having perhaps different m's), and actually one section with MM' arms requires 2 more capacitors and 2 more inductors than two lattice sections with m-derived arms.

It has been assumed previously that all four arms of the lattice were of the same pass-type (i.e., high-pass, low-pass, band-pass, etc.) and had the same cutoff frequencies. An interesting problem is the effect of different cutoff frequencies for the shunt and series arms. Previously a single frequency saw all four arms to be resistances, or the shunt and series arms inverse reactances. With different cutoff frequencies, however, there will be frequencies at which the shunt (or series) arms appear to be resistances while, at the same frequency, the other arms appear to be reactances. Let us discuss this problem for the following assumptions:

a. The series arms are the low-pass tee type, terminated, and contain L_1 and C_1.

b. The shunt arms are the low-pass pi type, terminated, and contain L_2 and C_2.

The cutoff frequencies of the shunt and series arms are different and are related by

(22) $\omega_{c2} = k\,\omega_{c1}$

Now (23) $\omega_{c1} = \frac{2}{\sqrt{L_1C_1}}$ and $\omega_{c2} = \frac{2}{\sqrt{L_2C_2}}$

∴ (24) $\frac{2}{\sqrt{L_1C_1}} = \frac{2}{k\sqrt{L_2C_2}}$

∴ (25) $L_2C_2 = \frac{1}{k^2} L_1C_1$

If we assume $k<1$, below ω_{c2} both arms are resistances. Between ω_{c1} and ω_{c2}, the series arms appear to be resistances and the shunt arms reactances. Above ω_{c1} all arms appear to be reactances (shunt and series opposite).

Let us define

(26) $y \equiv \sqrt{\frac{Z_a}{Z_b}}$

Below ω_{c2}

(27) $y = \sqrt{\frac{Z_a}{Z_b}} = \sqrt{\frac{K_1\sqrt{1+P_1}}{\frac{K_2}{\sqrt{1+P_2}}}} = \sqrt{\frac{K_1}{K_2}}\sqrt{\sqrt{1+P_1}\,\sqrt{1+P_2}}$

$$= \sqrt{\frac{K_1}{K_2}}\,(1+P_1)^{1/4}(1+P_2)^{1/4} = \left(1-\tfrac{1}{4}\omega^2L_1C_1\right)^{1/4}\left(1-\tfrac{1}{4k^2}\omega^2L_1C_1\right)^{1/4}$$

by assuming $\sqrt{\frac{K_1}{K_2}}=1$, with no loss in generality, other than a constant factor in particular. In this region the attenuation is

(28) $\tanh\left(\frac{\alpha}{2}\right) = \left(1-\frac{1}{4}\omega^2L_1C_1\right)^{1/4}\left(1-\frac{1}{4k^2}\omega^2L_1C_1\right)^{1/4}$

and we see the attenuation is infinite at $\omega=0$ and zero at ω_{c2}.

The iterative impedance in this region is

(29) $Z_i = \sqrt{Z_bZ_a} = \sqrt{K_1\sqrt{1+P_1}\,\frac{K_2}{\sqrt{1+P_2}}} = \sqrt{K_1K_2}\sqrt{\frac{\sqrt{1+P_1}}{\sqrt{1+P_2}}} = \sqrt{K_1K_2}\sqrt{\frac{\sqrt{1-\frac{1}{4}\omega^2L_1C_1}}{\sqrt{1-\frac{1}{4k^2}\omega^2L_1C_1}}}$

The iterative impedance is $\sqrt{K_1K_2}$ at $\omega=0$ and infinite at ω_{c2}.

In the region $\omega_{c2}<\omega<\omega_{c1}$ the series arms appear resistive, the shunt arms reactive, and the reactance is a negative[#3] reactance. Here

(30) $y = \sqrt{\frac{Z_a}{Z_b}} = \sqrt{\frac{Z_a}{-j|Z_b|}} = \sqrt{j\,\frac{Z_a}{|Z_b|}}$

$$= \sqrt{j}\,\sqrt{\left|\frac{Z_a}{Z_b}\right|} \equiv \frac{y_0}{\sqrt{2}} + j\,\frac{y_0}{\sqrt{2}}$$

In this region

(31) $$Z_L = \sqrt{Z_a Z_b} = \sqrt{K_1 K_2}\sqrt{\frac{\sqrt{1+\rho_1}}{\sqrt{1+\rho_2}}} = \sqrt{K_1 K_2}\sqrt{\frac{\sqrt{1-\frac{1}{4}\omega^2 L_1 C_1}}{j\sqrt{\frac{1}{4k^2}\omega^2 L_1 C_1 - 1}}}$$
$$= \sqrt{K_1 K_2}\left(\frac{1-\frac{1}{4}\omega^2 L_1 C_1}{\frac{1}{4k^2}\omega^2 L_1 C_1 - 1}\right)^{1/4} \angle -45^\circ$$

Thus here the iterative impedance is a complex number, always having an angle of -45°, and varying in magnitude from ∞ at ω_{c2} to zero at ω_{c1}. Returning to y we see that it is a complex number and hence the attenuation is not given by (3) or (4), and therefore the formula for attenuation must be developed for this case.

The transfer function for any lattice is *2,p379

(32) $$\gamma = \alpha + j\beta = 2\tanh^{-1}\sqrt{\frac{Z_b}{Z_a}} = \ln\left(\frac{\sqrt{\frac{Z_a}{Z_b}}+1}{\sqrt{\frac{Z_a}{Z_b}}-1}\right) = \ln\left(\frac{y+1}{y-1}\right)$$

Since it is more difficult to find the real part of the $\tanh^{-1}$ of a complex number, the log form will be used.

(33) $$\gamma = \ln\left(\frac{y+1}{y-1}\right) = \ln\left(\frac{\frac{y_0}{\sqrt{2}}+1+j\frac{y_0}{\sqrt{2}}}{\frac{y_0}{\sqrt{2}}-1+j\frac{y_0}{\sqrt{2}}}\right)$$
$$= \ln\left(\frac{\left(\frac{y_0}{\sqrt{2}}+1+j\frac{y_0}{\sqrt{2}}\right)\left(\frac{y_0}{\sqrt{2}}-1-j\frac{y_0}{\sqrt{2}}\right)}{\left(\frac{y_0}{\sqrt{2}}-1\right)^2+\left(\frac{y_0}{\sqrt{2}}\right)^2}\right) = \ln\left(\frac{\frac{y_0^2}{2}-1+\frac{y_0^2}{2}+j\left(\frac{y_0^2}{2}-\frac{y_0}{\sqrt{2}}-\frac{y_0^2}{2}-\frac{y_0}{\sqrt{2}}\right)}{\frac{y_0^2}{2}-2\frac{y_0}{\sqrt{2}}+1+\frac{y_0^2}{2}}\right)$$
$$= \ln\left(\frac{y_0^2-1+j\left(-\sqrt{2}\,y_0\right)}{y_0^2-\sqrt{2}\,y_0+1}\right) = \ln\left(\frac{y_0^2-1}{y_0^2-\sqrt{2}\,y_0+1}+j\,\frac{-\sqrt{2}\,y_0}{y_0^2-\sqrt{2}\,y_0+1}\right)$$

Now if

(34) $$\alpha + j\beta = \ln(A+jB)$$

Then (35) $$\alpha = \ln\sqrt{A^2+B^2} = \frac{1}{2}\ln\left(A^2+B^2\right)$$ (See *5)

Therefore in this case

$$(36)\ \alpha = \frac{1}{2}\ln\left[\frac{(y_0^2-1)^2+(-\sqrt{2}y_0)^2}{(y_0^2-\sqrt{2}y_0+1)^2}\right]$$

$$= \frac{1}{2}\ln\left[\frac{y_0^4-2y_0^2+1+2y_0^2}{(y_0^2-\sqrt{2}y_0+1)^2}\right] = \frac{1}{2}\ln\left[\frac{y_0^4+1}{(y_0^2-\sqrt{2}y_0+1)^2}\right]$$

Therefore when $y_0=\infty$, $\alpha=0$ and

when $y_0=0$, $\alpha=0$. Therefore there must be some value of y_0 that causes the maximum attenuation in this range. In Appendix D it is proved that this value is $y_0=1$. Therefore the maximum attenuation in this range is

$$(37)\ \alpha = \frac{1}{2}\ln\left[\frac{1+1}{(1-\sqrt{2}+1)^2}\right] = \frac{1}{2}\ln\left[\frac{2}{(2-\sqrt{2})^2}\right] = .88 \text{ nepers} = 7.64 \text{ db.}$$

The actual value of 7.64 db. will be reached only for very large or small k's, however, as y_0 does not necessarily pass through the value of 1.

The attenuation above ω_{c1} will be zero, as the shunt and series arms will then appear to be opposite reactances.

Plate 2 illustrates the attenuation of the lattice filter when (a) $\omega_{c2}=\omega_{c1}$, (b) $\omega_{c2}=.9\omega_{c1}$ and (c) $\omega_{c2}=.5\omega_{c1}$.

We have yet to determine the iterative impedance above ω_{c1}. The iterative impedance in this range is

$$(38)\ Z_L = \sqrt{Z_a Z_b} = \sqrt{K_1K_2}\left(\frac{\frac{1}{4}\omega^2L_1C_1-1}{\frac{1}{4k^2}\omega^2L_1C_1-1}\right)$$

Fig. 8 is a sketch of the iterative impedance. The real and imaginary parts are equal in the complex region.

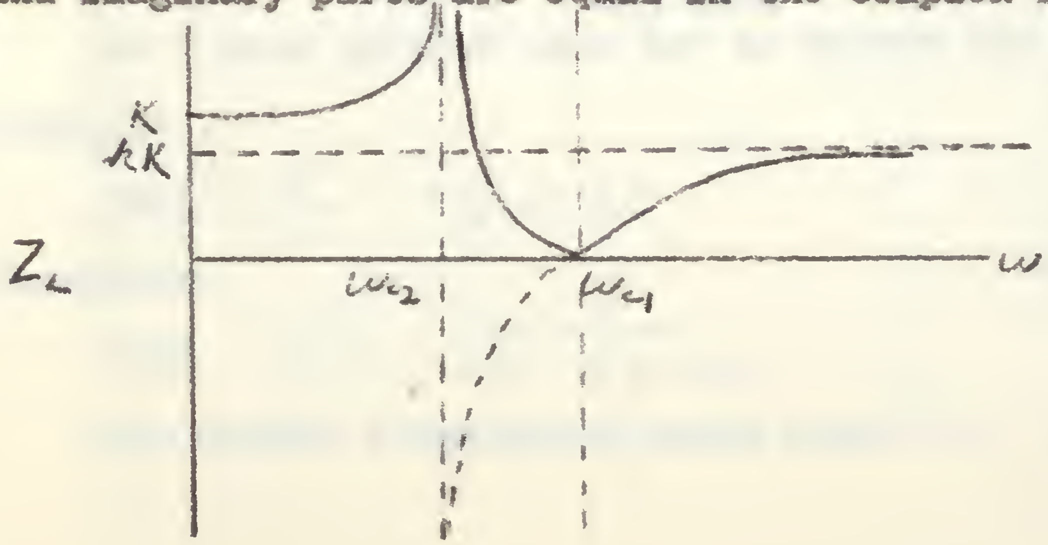


Fig. 8

Let us now investigate the filter formed when the arms are arranged as a tee rather than the lattice of previous discussion. This case is more complicated than the lattice because a tee structure is not an all-pass structure when series and shunt arms are opposite reactances. The tee will attenuate as did the lattice when all arms appear to be resistances; but for a portion of the frequency spectrum when arms are reactances, the filter will attenuate again. There also will be a portion of the frequency spectrum wherein the reactances combine to give a resistive iterative impedance with no attenuation. There will be attenuation when $\rho > 0$ or $\rho < -1$. Attenuation will be zero when $0 > \rho > -1$.*3,p296 With constant-K type arms we can have the arrangement of either fig. 9 or fig. 10.

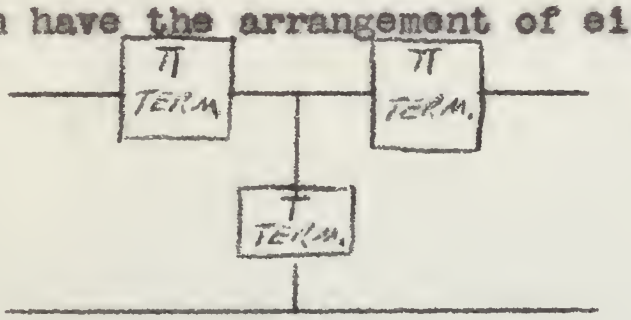


Fig. 9

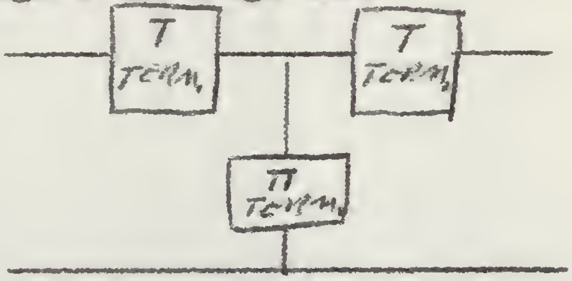


Fig. 10

The results will not be the same for the two figures.

First let us assume the arrangement of fig. 9. Then

$$(39)\quad \rho = \frac{2Z_\pi}{4Z_T} = \frac{\frac{2K}{\sqrt{1+\rho_0}}}{4K\sqrt{1+\rho_0}} = \frac{1}{2(1+\rho_0)}$$

As a more special case let us assume the arms are low-pass. From (9)

$$(40)\quad \rho_0 = -\frac{1}{4}\omega^2 LC$$

Therefore

$$(41)\quad \rho = \frac{1}{2(1-\frac{1}{4}\omega^2 LC)}$$

The cutoff frequencies exist when $\rho = 0$ and $\rho = -1$.

When

(42) $0 = \frac{1}{2(1-\frac{1}{4}\omega^2LC)}$

we see $\omega=\infty$ is a cutoff frequency.

When

(43) $-1 = \frac{1}{2(1-\frac{1}{4}\omega^2LC)}$, $\omega = \frac{\sqrt{6}}{\sqrt{LC}}$

Below $\omega=\frac{2}{\sqrt{LC}}$, all arms appear resistive, therefore ρ is positive in this range and then *3,p297

(44) $\alpha = 2\sinh^{-1}\sqrt{\rho} = 2\sinh^{-1}\sqrt{\frac{1}{2(1-\frac{1}{4}\omega^2LC)}}$

When $\omega\to 0, \alpha = 2\sinh^{-1}\sqrt{\frac{1}{2}} = 11.4$ db

*3,p297

For $\frac{2}{\sqrt{LC}} > \omega > \frac{\sqrt{6}}{\sqrt{LC}}, \rho < -1$ and then

(45) $\alpha = 2\sinh^{-1}\sqrt{-1-\rho} = 2\sinh^{-1}\sqrt{-1-\frac{1}{2(1-\frac{1}{4}\omega^2LC)}}$

For $\omega > \frac{\sqrt{6}}{\sqrt{LC}}$, ρ is between 0 and -1 and the attenuation is zero.

*3,p286

The iterative impedance of our section is

(46) $Z = \sqrt{2Z_\pi Z_T + \frac{(2Z_\pi)^2}{4}} = \sqrt{\frac{2K}{\sqrt{1+P_0}}K\sqrt{1+P_0}+\frac{4K^2}{4(1+P_0)}} = K\sqrt{2+\frac{1}{1+P_0}}$

For low-pass arms,

(47) $Z = K\sqrt{2+\frac{1}{1-\frac{1}{4}\omega^2LC}}$

When $\omega=0$, $Z=K\sqrt{3}$. Further, $Z=0$ when $\omega=\frac{\sqrt{6}}{\sqrt{LC}}$. For large ω, $Z\to K\sqrt{2}$.

Figures 11 and 12 show the attenuation and iterative impedance for the tee arrangement of fig. 9, using low-pass arms.

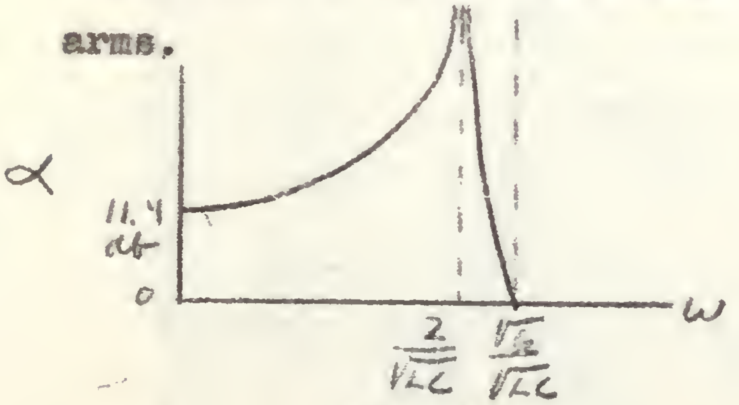


Fig. 11

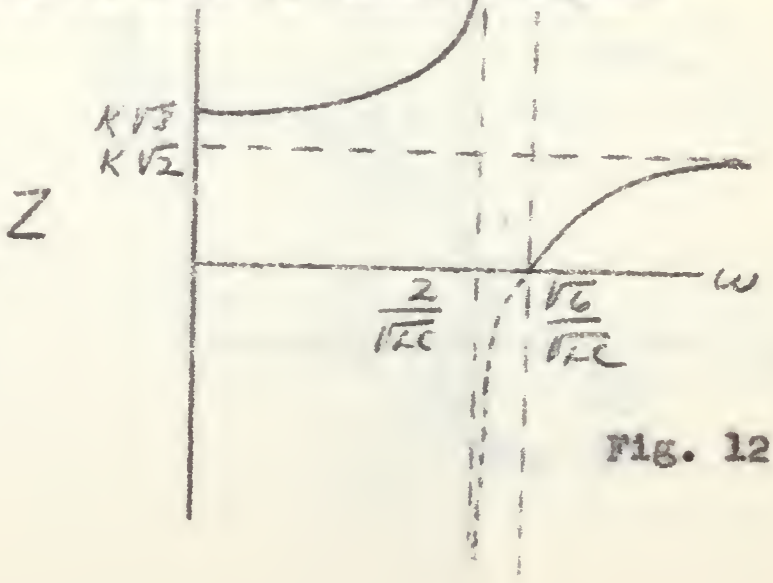


Fig. 12

As another special case using the arrangement of fig. 9 let us assume the arms are high-pass. From (11)

(48) $\rho_0 = -\frac{1}{4\omega^2 LC}$ Therefore

(49) $\rho = \frac{1}{2\left(1-\frac{1}{4\omega^2 LC}\right)}$

Cutoff frequencies exist when $\rho = 0$ and $\rho = -1$. When

(50) $0 = \frac{1}{2\left(1-\frac{1}{4\omega^2 LC}\right)}$, $\omega = 0$ and when

(51) $-1 = \frac{1}{2\left(1-\frac{1}{4\omega^2 LC}\right)}$, $\omega = \frac{1}{\sqrt{6LC}}$

Further we can see $\rho = \infty$ when $\omega = \frac{1}{2\sqrt{LC}}$

Above $\omega = \frac{1}{2\sqrt{LC}}$, ρ is positive and the attenuation in this range is

(52) $\alpha = 2\sinh^{-1}\rho = 2\sinh^{-1}\sqrt{\frac{1}{2\left(1-\frac{1}{4\omega^2 LC}\right)}}$

For the range $\frac{1}{\sqrt{6LC}} < \omega < \frac{1}{2\sqrt{LC}}$, $\rho < -1$ and the attenuation is

(53) $\alpha = 2\sinh^{-1}\sqrt{-1-\rho} = 2\sinh^{-1}\sqrt{-1-\frac{1}{2\left(1-\frac{1}{4\omega^2 LC}\right)}}$

For the range $\omega < \frac{1}{\sqrt{6LC}}$, $-1 < \rho < 0$ and the attenuation is 0.

The iterative impedance of the section is

(54) $Z = K\sqrt{2+\frac{1}{1+\rho_0}} = K\sqrt{2+\frac{1}{1-\frac{1}{4\omega^2 LC}}}$

When $\omega = 0$, $Z = K\sqrt{2}$ and for $\omega \to \infty$, $Z \to K\sqrt{3}$. For the range $\frac{1}{\sqrt{6LC}} < \omega < \frac{1}{2\sqrt{LC}}$ the iterative impedance is an inductive reactance.

Figures 13 and 14 show the attenuation and iterative impedance for the tee arrangement of fig. 9, using high-pass arms.

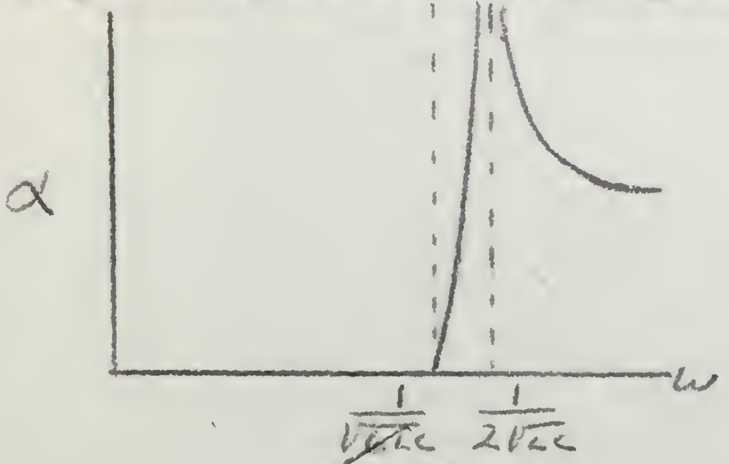


Fig. 13

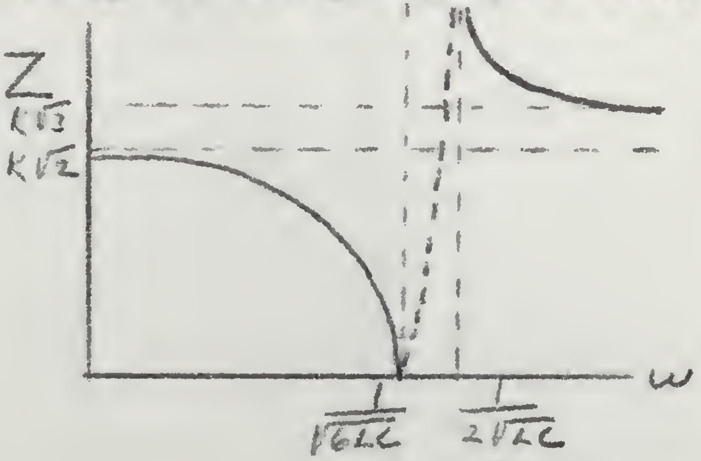


Fig. 14

Let us assume the arrangement of fig. 10. Then

$$(55)\quad \rho = \frac{2Z_T}{4Z_\pi} = \frac{2K\sqrt{1+\rho_0}}{\frac{4K}{\sqrt{1+\rho_0}}} = \frac{1}{2}(1+\rho_0)$$

As a more special case let us assume the arms are low-pass Therefore

$$(56)\quad \rho = \frac{1}{2}\left(1 - \frac{1}{4}\omega^2 LC\right)$$

The cutoff frequencies exist when $\rho = 0$ and $\rho = -1$. When

$$(57)\quad 0 = \frac{1}{2}\left(1 - \frac{1}{4}\omega^2 LC\right)$$

we see $\omega = \frac{2}{\sqrt{LC}}$ is a cutoff frequency. When

$$(58)\quad -1 = \frac{1}{2}\left(1 - \frac{1}{4}\omega^2 LC\right)$$

we see $\omega = \sqrt{\frac{12}{LC}}$ is a cutoff frequency. Below $\omega = \frac{2}{\sqrt{LC}}$ all arms are resistive, hence ρ is positive and

$$(59)\quad \alpha = 2\sinh^{-1}\sqrt{\rho} = 2\sinh^{-1}\sqrt{\tfrac{1}{2}\left(1 - \tfrac{1}{4}\omega^2 LC\right)}$$

For the range $\frac{2}{\sqrt{LC}} < \omega < \sqrt{\frac{12}{LC}}$, $-1 < \rho < 0$ and the attenuation is 0. For the range $\omega > \sqrt{\frac{12}{LC}}$, $\rho < -1$ and

$$(60)\quad \alpha = 2\sinh^{-1}\sqrt{-1-\rho} = 2\sinh^{-1}\sqrt{-1-\tfrac{1}{2}\left(1 - \tfrac{1}{4}\omega^2 LC\right)}$$

The iterative impedance of our section is

$$(61)\quad Z = \sqrt{2Z_\pi Z_T + \frac{(2Z_T)^2}{4}} = \sqrt{2K^2 + K^2(1+\rho_0)} = K\sqrt{3+\rho_0} = K\sqrt{3 - \tfrac{1}{4}\omega^2 LC}$$

Figures 15 and 16 show the attenuation and iterative impedance for the arrangement of fig. 10, using low-pass arms.

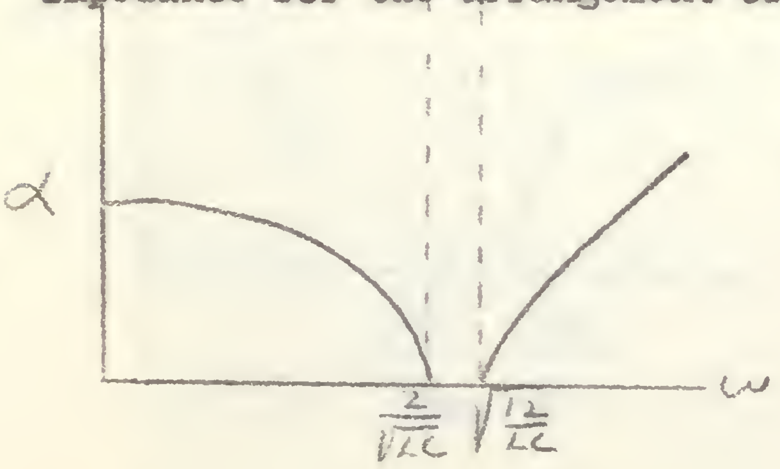


Fig. 15

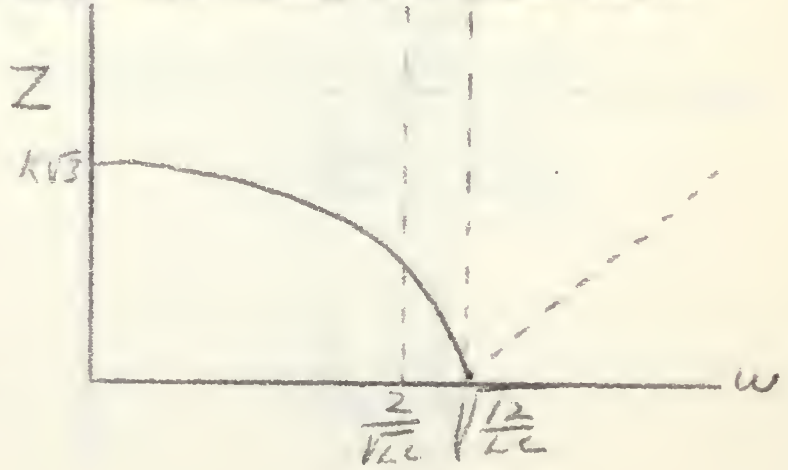


Fig. 16

As another special case using the arrangement of fig. 10 let us assume the arms are high-pass. Then

(62) $\rho = \frac{1}{2}(1+\rho_0) = \frac{1}{2}\left(1 - \frac{1}{4\omega^2 LC}\right)$

The cutoff frequencies exist when $\rho = 0$ and $\rho = -1$. When

(63) $0 = \frac{1}{2}\left(1 - \frac{1}{4\omega^2 LC}\right)$, $\omega = \frac{1}{2\sqrt{LC}}$. When

(64) $-1 = \frac{1}{2}\left(1 - \frac{1}{4\omega^2 LC}\right)$, $\omega = \frac{1}{\sqrt{12LC}}$.

For the range $\omega < \frac{1}{\sqrt{12LC}}$, $\rho < -1$ and

(65) $\alpha = 2\sinh^{-1}\sqrt{-1-\rho} = 2\sinh^{-1}\sqrt{-1-\frac{1}{2}\left(1-\frac{1}{4\omega^2 LC}\right)}$

For the range $\frac{1}{\sqrt{12LC}} < \omega < \frac{1}{2\sqrt{LC}}$, $-1 < \rho < 0$ and the attenuation is 0. For the range $\omega > \frac{1}{2\sqrt{LC}}$, the arms are resistive, ρ is positive, and

(66) $\alpha = 2\sinh^{-1}\sqrt{\rho} = 2\sinh^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{4\omega^2 LC}\right)}$

The iterative impedance of the section is

(67) $Z = K\sqrt{3+\rho_0} = K\sqrt{3-\frac{1}{4\omega^2 LC}}$

Figures 17 and 18 show the attenuation and iterative impedance for the arrangement of fig. 10, using high-pass arms.

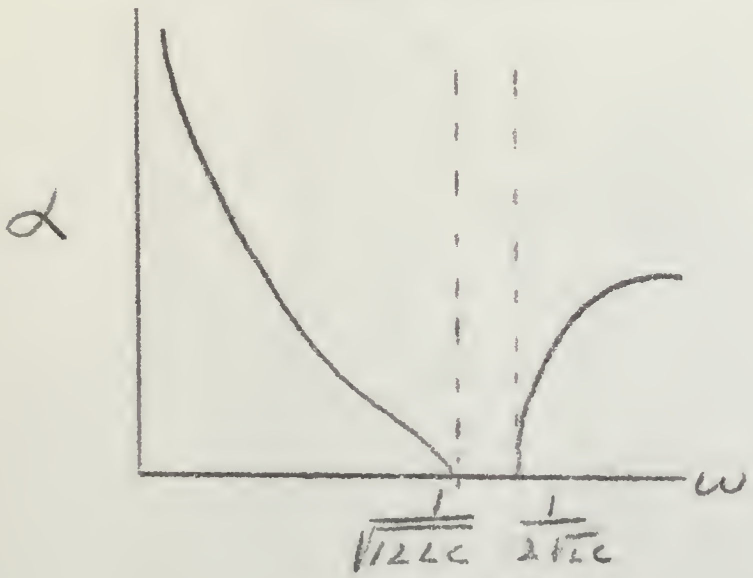


Fig. 17

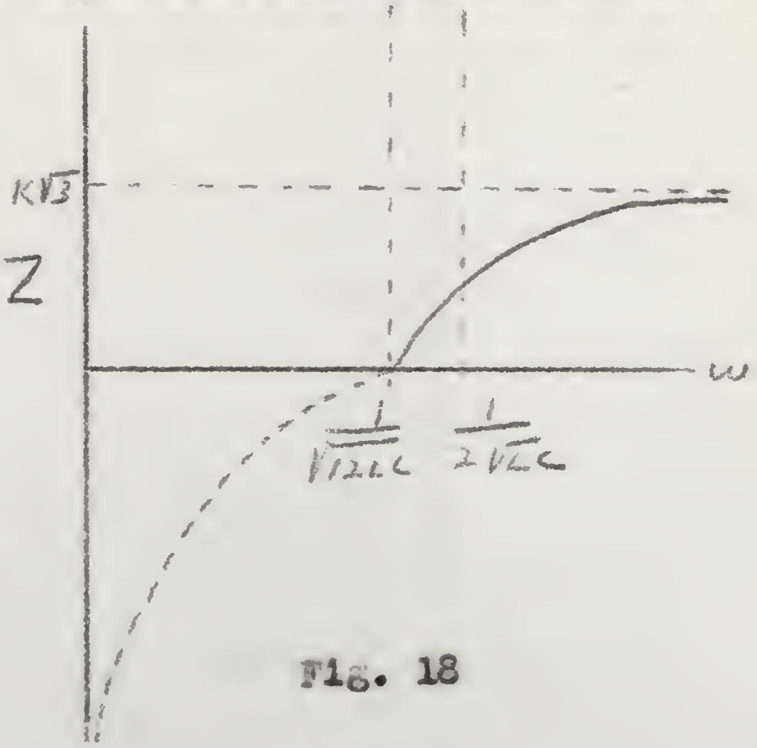


Fig. 18

Since for the lattice we investigated the effect of using m-derived terminated arms, let us now for completeness sake investigate the effect of m-derived arms in a tee arrangement. We can have the arrangement of either fig. 19 or fig. 20.

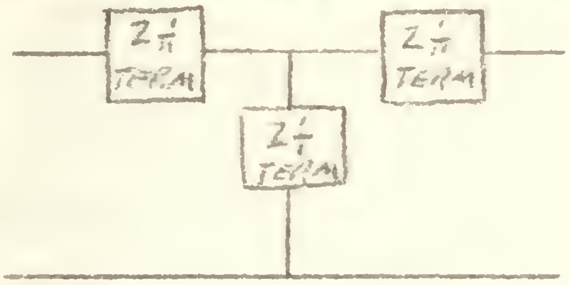


Fig. 19

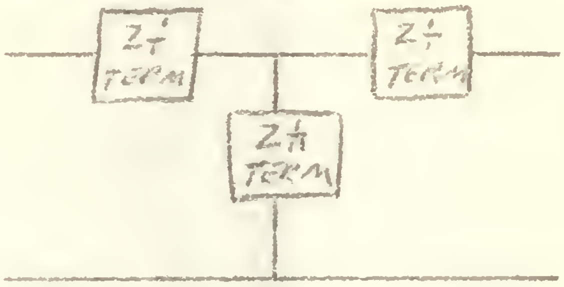


Fig. 20

For the case of fig. 19, using (14) and (15),

$$(68)\quad \rho = \frac{2Z_\pi'}{4Z_T'} = \frac{\dfrac{K[1+(1-m^2)\rho_0]}{\sqrt{1+\rho_0}}}{\dfrac{2K\sqrt{1+\rho_0}}{[1+(1-m^2)\rho_0]}} = \frac{[1+(1-m^2)\rho_0]^2}{2(1+\rho_0)}$$

$$(69)\quad Z = K\sqrt{2(1+\rho_0)} = K\sqrt{2+\frac{[1+(1-m^2)\rho_0]^2}{1+\rho_0}}$$

As a special case of fig. 19 let us assume the arms are low-pass. Using (9) then

$$(70)\quad \rho = \frac{[1+(1-m^2)(-\frac{1}{4}\omega^2LC)]^2}{2(1-\frac{1}{4}\omega^2LC)}$$ and

$$(71)\quad Z = K\sqrt{2+\frac{[1+(1-m^2)(-\frac{1}{4}\omega^2LC)]^2}{1-\frac{1}{4}\omega^2LC}}$$

As a numerical example using m .6, it can be seen $Z=\infty$ when $\omega = \frac{2}{\sqrt{LC}}$ and ∞. $Z=0$ when $\omega = \frac{5.25}{\sqrt{LC}}$ or $\frac{2.05}{\sqrt{LC}}$.

Figures 21 and 22 show the attenuation and iterative impedance for the arrangement of fig. 19, using low-pass m-terminated arms, with m = .6.

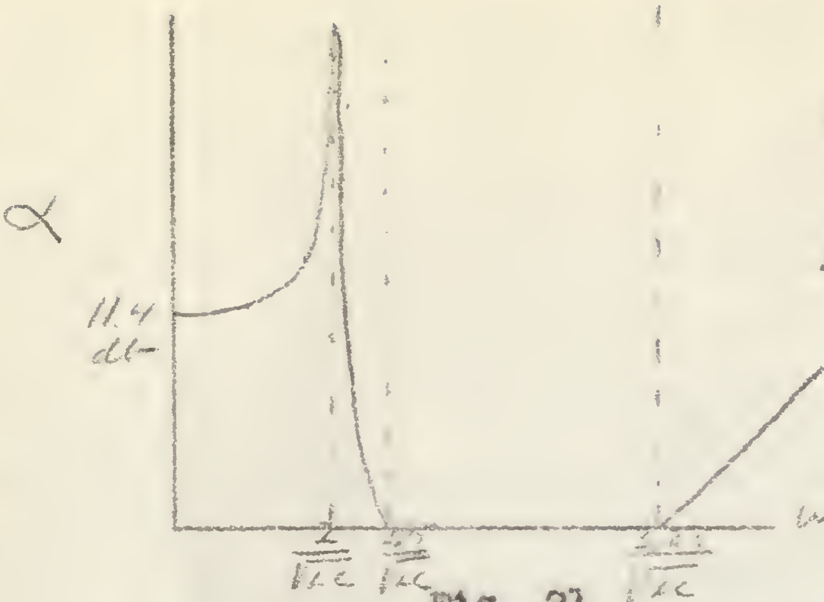


Fig. 21

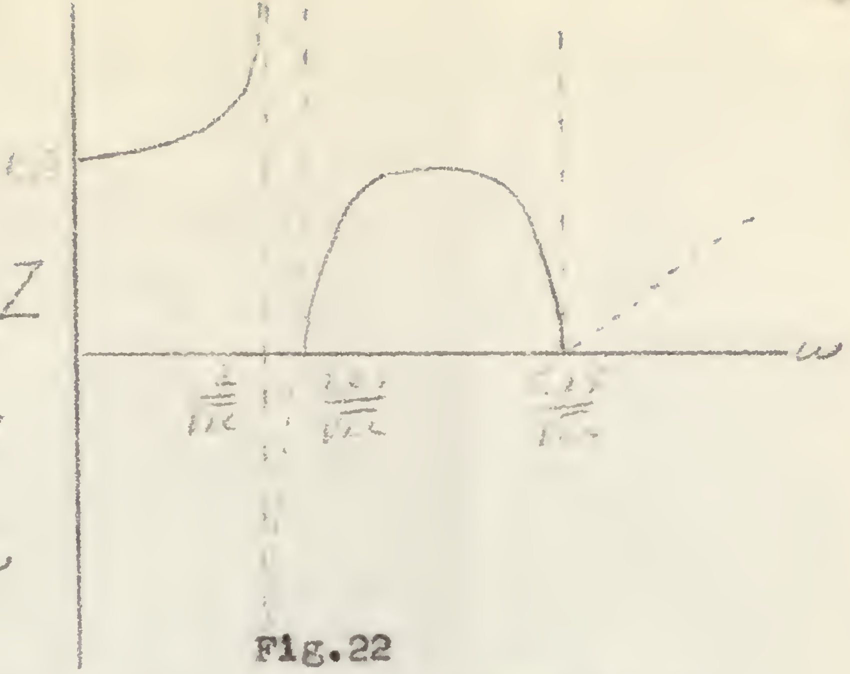


Fig.22

It should be noted the attenuation at $\omega = 0$ is the same as in fig. 11, but here we have a band-pass rather than the high-pass filter of fig. 11.

Let us consider the case of fig. 19 when the arms are high-pass. Using (11), (68), and (69),

$$(72)\quad \rho = \frac{\left[1 + (1-m^2)\left(-\frac{1}{4\omega^2LC}\right)\right]^2}{2\left(1 - \frac{1}{4\omega^2LC}\right)}$$

$$(73)\quad Z = K\sqrt{2 + \frac{\left[1 + (1-m^2)\left(-\frac{1}{4\omega^2LC}\right)\right]^2}{1 - \frac{1}{4\omega^2LC}}}$$

As a numerical example using m = .6 it can be seen $Z = \infty$ when $\omega = \frac{1}{2\sqrt{LC}}$ and 0. $Z = 0$ when $\omega = \frac{1}{2.05\sqrt{LC}}$ and $\frac{1}{1.25\sqrt{LC}}$.

Figures 23 and 24 show the attenuation and iterative impedance for the arrangement of fig. 19, using high-pass m-terminated arms, with m = .6.

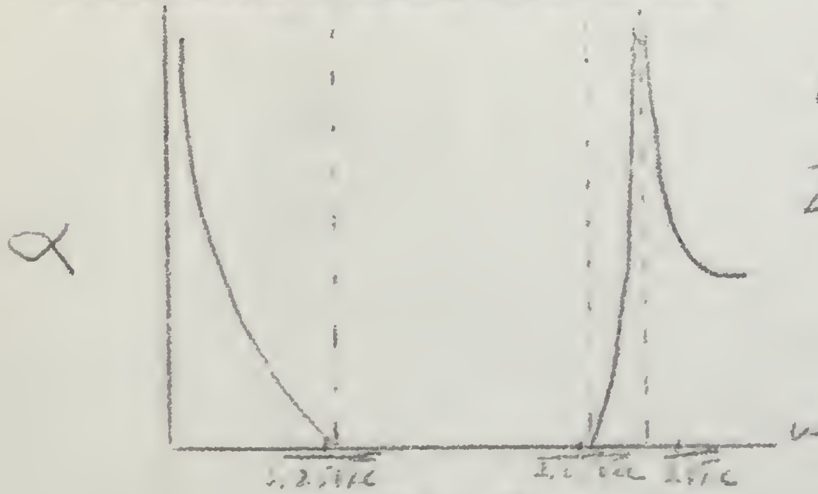


Fig. 23

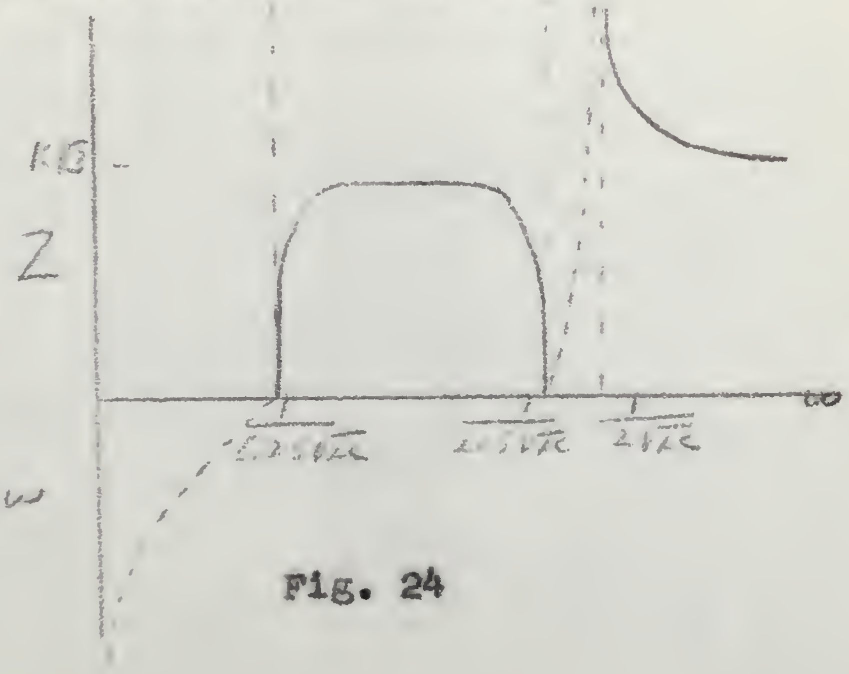


Fig. 24

For the case of fig. 20, using (14) and (15)

$$(74)\quad \rho = \frac{2Z_T'}{4Z_\pi'} = \frac{\dfrac{k\sqrt{1+\rho_0}}{[1+(1-m^2)\rho_0]}}{\dfrac{2k[1+(1-m^2)\rho_0]}{\sqrt{1+\rho_0}}} = \frac{1+\rho_0}{2[1+(1-m^2)\rho_0]^2}$$

$$(75)\quad Z = k\sqrt{2(1+\rho)} = k\sqrt{2+\frac{1+\rho_0}{[1+(1-m^2)\rho_0]^2}}$$

As a special case of fig. 20 let us assume the arms are low-pass. Then

$$(76)\quad \rho = \frac{1-\frac{1}{4}\omega^2LC}{2[1+(1-m^2)(-\frac{1}{4}\omega^2LC)]^2}$$

$$(77)\quad Z = k\sqrt{2+\frac{1-\frac{1}{4}\omega^2LC}{[1+(1-m^2)(-\frac{1}{4}\omega^2LC)]^2}}$$

As a numerical example using m .6 it can be seen $Z=\infty$ when $\omega=\frac{2.5}{\sqrt{LC}}$ and $Z=0$ when $\omega=\frac{2.14}{\sqrt{LC}}$ or $\frac{2.9}{\sqrt{LC}}$.

Figures 25 and 26 show the attenuation and iterative impedance for the arrangement of fig. 20, using low-pass m-terminated arms, with m=.6.

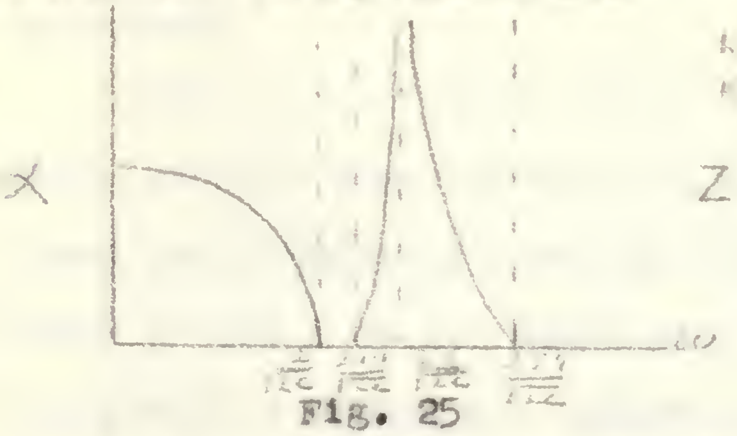


Fig. 25

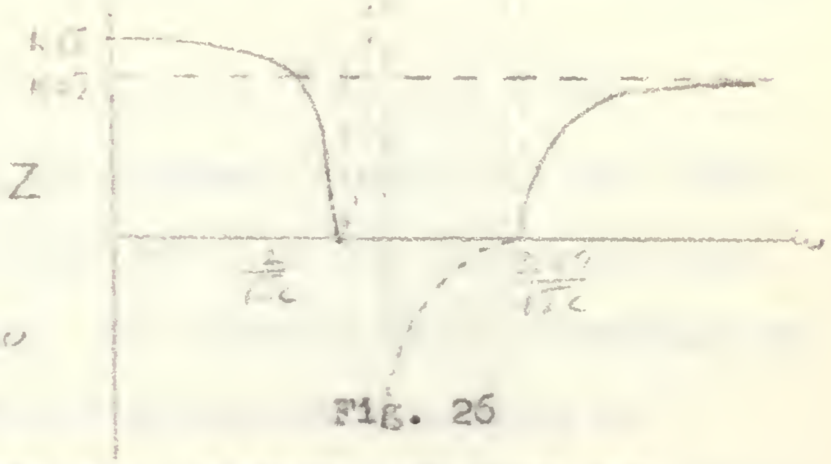


Fig. 26

Let us consider the case of fig. 20 when the arms are high-pass. Then

$$(78)\quad \rho = \frac{1-\frac{1}{4\omega^2LC}}{2\left[1+(1-m^2)(-\frac{1}{4\omega^2LC})\right]^2} \qquad \text{and}$$

$$(79)\quad Z = k\sqrt{2+\frac{1-\frac{1}{4\omega^2LC}}{\left[1+(1-m^2)(-\frac{1}{4\omega^2LC})\right]^2}}$$

As a numerical example using m=.6, it can be seen $Z=\infty$ when $\omega = \frac{1}{2.5\sqrt{LC}}$ and $Z=0$ when $\omega = \frac{1}{2.14\sqrt{LC}}$ or $\frac{1}{3.59\sqrt{LC}}$.

Figures 27 and 28 show the attenuation and iterative impedance for the arrangement of fig. 20, using high-pass m-terminated arms, with m=.6.

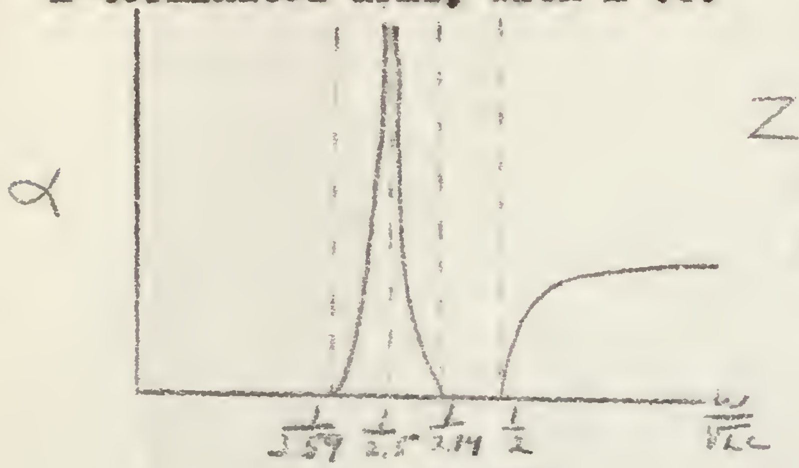


Fig. 27

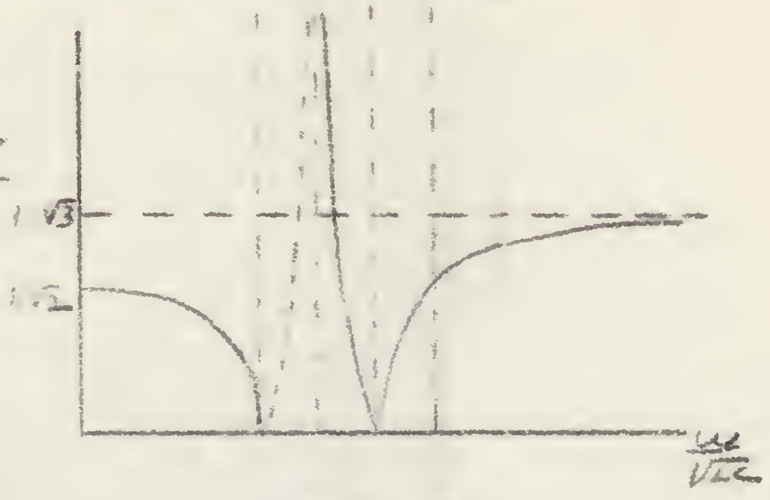


Fig. 28

For all tee type filters discussed thus far it is to be noted that at one end of the attenuation band the attenuation approaches

$$(80)\quad \alpha = 2\sinh^{-1}\sqrt{\rho} \rightarrow 2\sinh^{-1}\sqrt{\tfrac{1}{2}} = 11.4\ db$$

which is much lower than is to be desired. However in all these cases the assumption was made that the K of the series arm was equal to the K of the shunt arm. By removing this assumption we can obtain a method for making the attenuation as high as desired at one end of the attenuation band. Defining the series arms as arms 1 and the shunt arm as arm 2, let us again investigate the attenuation of the arrangement of fig. 9. Now

$$(81)\quad \rho = \frac{2Z_\pi}{4Z_T} = \frac{\frac{2K_1}{\sqrt{1+\rho_1}}}{4K_2\sqrt{1+\rho_2}} = \frac{1}{2}\frac{K_1}{K_2}\frac{1}{\sqrt{1+\rho_1}\sqrt{1+\rho_2}}$$

In that part of the attenuation band in which we are presently interested in increasing the attenuation, the arms appear as resistances and the attenuation is

(44) $\alpha = 2 \sinh^{-1} \sqrt{\rho}$

Therefore to increase α in this range we want to increase ρ as ω approaches 0 for the high-pass filter or as ω approaches ∞ for the low-pass filter. For the high-pass filter (low-pass arms)

(82) $\rho = \frac{1}{2} \frac{K_1}{K_2} \frac{1}{\sqrt{1-\frac{1}{4}\omega^2 L_1 C_1}\sqrt{1-\frac{1}{4}\omega^2 L_2 C_2}}$

and as $\omega \to 0$, $\rho \to \frac{1}{2}\frac{K_1}{K_2}$, and $\alpha \to 2 \sinh^{-1}\sqrt{\frac{1}{2}\frac{K_1}{K_2}}$

For the low-pass filter (high-pass arms)

(83) $\rho = \frac{1}{2} \frac{K_1}{K_2} \frac{1}{\sqrt{1-\frac{1}{4\omega^2 L_1 C_1}-\frac{1}{4\omega^2 L_2 C_2}}}$

and as $\omega \to \infty$, $\rho \to \frac{1}{2}\frac{K_1}{K_2}$, and $\alpha \to 2 \sinh^{-1}\sqrt{\frac{1}{2}\frac{K_1}{K_2}}$ as for the high-pass filter case. Therefore by determining the ratio $\frac{K_1}{K_2}$ we can make α approach whatever limit we please as $\omega \to 0$, or as $\omega \to \infty$, as the case may be. Now

(84) $K_1 = \sqrt{\frac{L_1}{C_1}}$ and $K_2 = \sqrt{\frac{L_2}{C_2}}$

but we are not completely independent in our choice of L's and C's as we still must meet the requirement that the cutoff frequencies must be the same. For the low-pass arm case

(85) $\omega_{c_1} = \omega_{c_2} = \frac{2}{\sqrt{L_1 C_1}} = \frac{2}{\sqrt{L_2 C_2}}$ or $L_1 C_1 = L_2 C_2$

For the high-pass arm case

(86) $\omega_{c_1} = \omega_{c_2} = \frac{1}{2\sqrt{L_1 C_1}} = \frac{1}{2\sqrt{L_2 C_2}}$ or $L_1 C_1 = L_2 C_2$

as before. These are the only conditions the L's and C's must meet, however (other than the desired impedance values), hence even fixing (86) we can pick any arbitrary values for (84) providing we accept the impedance characteristics.

Investigating the impedance for this case, instead of (46) we have

(87) $Z = \sqrt{2Z_\pi Z_T + \frac{(2Z_\pi)^2}{4}} = \sqrt{2\frac{K_1}{\sqrt{1+\rho_1}}K_2\sqrt{1+\rho_2} + \frac{4K_1^2}{4(1+\rho_1)}} = \sqrt{2K_1K_2\frac{\sqrt{1+\rho_2}}{\sqrt{1+\rho_1}} + \frac{K_1^2}{1+\rho_1}}$

Because of (85)

(88) $\rho_1 = \rho_2$ Therefore

(89) $Z = \sqrt{2K_1K_2 + \frac{K_1^2}{1+\rho_1}} = \sqrt{K_1K_2}\sqrt{2 + \frac{\frac{K_1}{K_2}}{1+\rho_1}}$, rather than

(46) $Z = K\sqrt{2 + \frac{1}{1+\rho_0}}$

For the low-pass arm (high-pass filter) case then, the impedance curve will still have the general outline of fig. 12 but the limiting values will be different. The value $\sqrt{K_1K_2}$ could be chosen independently of $\frac{K_1}{K_2}$ if (85) did not apply. Let us now investigate the effect of different cutoff frequencies.

Let us assume the arrangement of fig. 9 to investigate the effect of different cutoff frequencies of the shunt and series arms of the tee type filter. Define a series arm as arm 1 and a shunt arm as arm 2. Assume $\omega_{c2} = k\,\omega_{c1}$, where k is a constant. When $k = 1$ previous argument would hold. Now

(90) $\omega_{c2} = \frac{2}{\sqrt{L_2C_1}} = k\,\omega_{c1} = \frac{2k}{\sqrt{L_2C_2}}$. Therefore

(91) $L_2C_2 = k^2 L_1C_1$

For convenience of explanation assume $k > 1$ or $\omega_{c2} > \omega_{c1}$. Those frequencies $\omega < \omega_{c1}$ see fig. 29. Those frequencies $\omega_{c1} < \omega < \omega_{c2}$ see fig. 30. Those frequencies $\omega_{c2} < \omega$ see fig. 31. It is obvious fig. 29 will cause attenuation. Fig. 30 will cause attenuation in some manner we have yet to determine. Fig. 31 is a high-pass section, so we suspect it will attenuate for some frequencies and pass the rest.

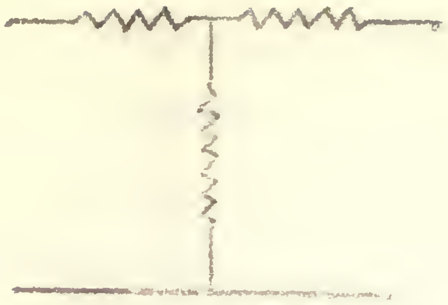


Fig. 29

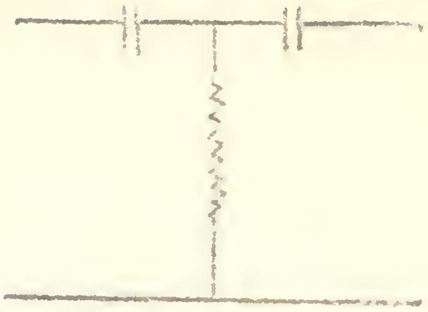


Fig. 30

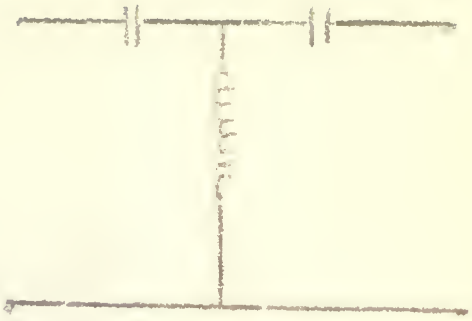


Fig. 31

(92) $\rho = \frac{2Z_s}{4Z_T} = \frac{2\frac{K_1}{\sqrt{1+\rho_1}}}{4K_2\sqrt{1+\rho_2}} = \frac{1}{2}\frac{K_1}{K_2}\frac{1}{\sqrt{1+\rho_1}\sqrt{1+\rho_2}}$

For simplicity assume $\frac{K_1}{K_2} = 1$. Now

(93) $\rho_1 = -\frac{1}{4}\omega^2 L_1 C_1$ and

(94) $\rho_2 = -\frac{1}{4}\omega^2 L_2 C_2 = -\frac{1}{4k^2} L_1 C_1$

Therefore

(95) $\rho = \frac{1}{2\sqrt{1-\frac{1}{4}\omega^2 L_1 C_1}\sqrt{1-\frac{1}{4k^2}\omega^2 L_1 C_1}}$

In the range $\omega < \omega_{c_1}$, ρ is a positive number and

(96) $\alpha = 2\sinh^{-1}\sqrt{\rho} = 2\sinh^{-1}\sqrt{\frac{1}{2\sqrt{1-\frac{1}{4}\omega^2 L_1 C_1}\sqrt{1-\frac{1}{4k^2}\omega^2 L_1 C_1}}}$

In the range $\omega_{c_1} < \omega < \omega_{c_2}$, $\boldsymbol{\rho}$ is a complex number, actually

(97) $\boldsymbol{\rho} = \frac{1}{2 \times \text{real} \times \text{imaginary}}$ = pure imaginary

The heavy $\boldsymbol{\rho}$ will designate the fact that it is not a real number, and the light ρ will designate the modulus of $\boldsymbol{\rho}$.

In the complex case the attenuation for the tee arrangement is [*3,p 302]

(98) $\alpha = \cosh^{-1}\left[\sqrt{(\rho-1)^2+4\rho\cos^2\frac{\varphi}{2}} + \rho\right]$

where φ is the angle of $\boldsymbol{\rho}$.

In our case φ is 90° hence (98) reduces to

(99) $\alpha = \cosh^{-1}\left[\sqrt{\rho^2+1} + \rho\right]$ and in this range

(100) $\rho = \frac{1}{2\sqrt{\frac{1}{4}\omega^2 L_1 C_1 - 1}\sqrt{1-\frac{1}{4k^2}\omega^2 L_1 C_1}}$

Above ω_{c_2}, ρ is a negative real number and we can return to the symbol ρ . Here

$$(101)\quad \rho = -\frac{1}{2}\,\frac{1}{\sqrt{\frac{1}{4}\omega^2 L_1 C_1 - 1}\,\sqrt{\frac{1}{4k^2}\omega^2 L_1 C_1 - 1}}$$

For any frequency

$$(102)\quad Z = \sqrt{2 Z_\pi Z_T + \frac{(2 Z_\pi)^2}{4}} = \sqrt{K_1 K_2}\sqrt{2\,\frac{\sqrt{1-\frac{1}{4k^2}\omega^2 L_1 C_1}}{\sqrt{1-\frac{1}{4}\omega^2 L_1 C_1}} + \frac{\frac{K_1}{K_2}}{1-\frac{1}{4}\omega^2 L_1 C_1}}$$

For $K_1 = K_2 = K$

$$(103)\quad Z = K\sqrt{2\,\frac{\sqrt{1-\frac{1}{4k^2}\omega^2 L_1 C_1}}{\sqrt{1-\frac{1}{4}\omega^2 L_1 C_1}} + \frac{1}{1-\frac{1}{4}\omega^2 L_1 C_1}}$$

As a numerical example, Plate 3 shows α vs. ω for the high-pass filter when $k=1$ and $k=2$. It is to be noted the new cutoff frequency of the shunt arm had no harmful effect, other than the shifting of the cutoff frequency of the filter to near the cutoff frequency of the shunt arm. Fig. 32 is a sketch of the iterative impedance for this case when $k=2$. The impedance is complex between ω_{c_1} and ω_{c_2}.

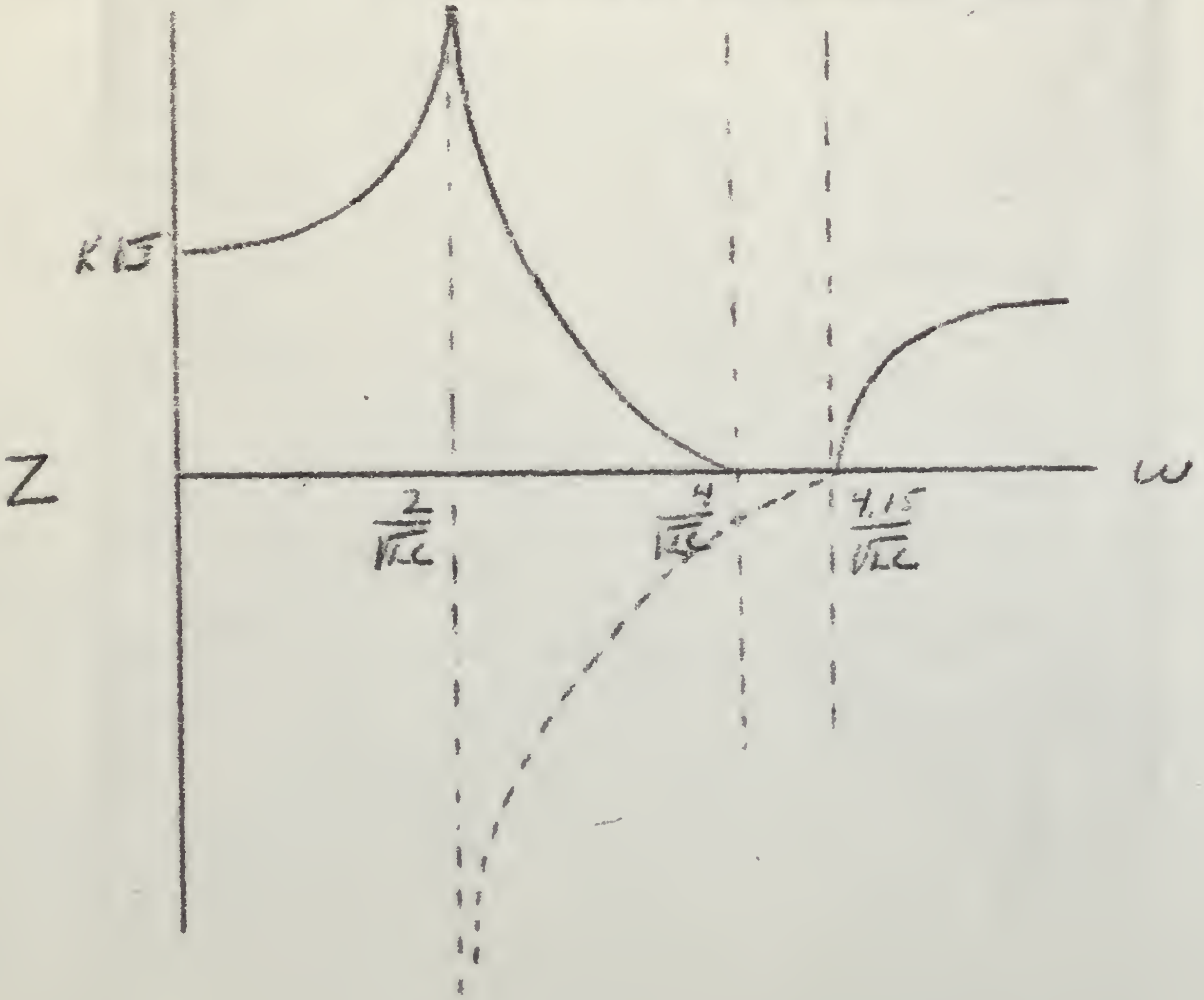


Fig. 32

To conclude the thesis three cases will be discussed to show, partly, why it was concluded arms of both the lattice and tee type filters should be all of the same pass-type.

For the first case let us assume we have a lattice (fig. 33) with the series (no. 2) arms high-pass with pi terminations and the shunt (no. 1) arm low-pass with tee termination. Assume the cutoff frequencies of the arms are the same, and the K's the same.

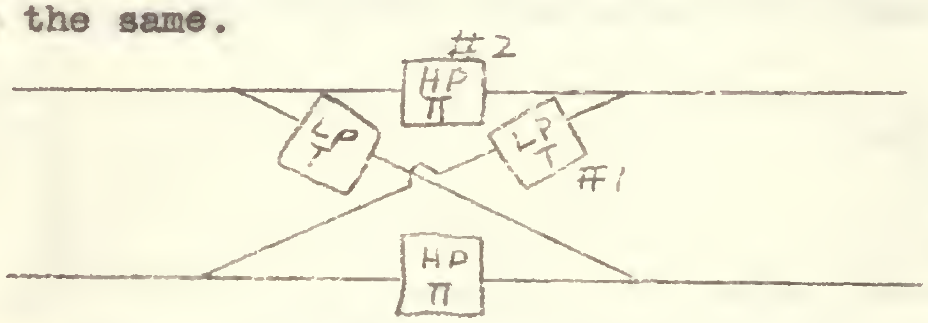


Fig. 33

(104) $Z_T = K\sqrt{1+\rho_1} = K\sqrt{1-\frac{1}{4}\omega^2 L_1 C_1}$

(105) $Z_\pi = \frac{K}{\sqrt{1+\rho_2}} = \frac{K}{\sqrt{1-\frac{1}{4\omega^2 L_2 C_2}}}$

For the cutoff frequencies to be the same,

(106) $\omega_{c_1}^2 = \frac{4}{L_1 C_1} = \omega_{c_2}^2 = \frac{1}{4 L_2 C_2}$ and

(107) $L_2 C_2 = \frac{L_1 C_1}{16}$

Therefore

(108) $y = \sqrt{\frac{Z_2}{Z_1}} = \left(1-\frac{1}{4}\omega^2 L_1 C_1\right)^{1/4}\left(1-\frac{1}{4\omega^2 \frac{L_1 C_1}{16}}\right)^{1/4}$

$= \left(1-\frac{1}{4}\omega^2 L_1 C_1\right)^{1/4}\left(1-\frac{4}{\omega^2 L_1 C_1}\right)^{1/4} = \left(2-\frac{1}{4}\omega^2 L_1 C_1-\frac{4}{\omega^2 L_1 C_1}\right)^{1/4}$

The quantity under the radical will always be negative. Therefore y is complex and its real and imaginary parts are equal. (30) and (36) apply.

(109) $y_0 = \left(\frac{1}{4}\omega^2 L_1 C_1 + \frac{4}{\omega^2 L_1 C_1} - 2\right)^{1/4}$

Therefore

(110) $\alpha = \frac{1}{2}\ln\left[\frac{\frac{1}{4}\omega^2 L_1 C_1 + \frac{4}{\omega^2 L_1 C_1} - 2 + 1}{\left[\left(\frac{1}{4}\omega^2 L_1 C_1 + \frac{4}{\omega^2 L_1 C_1} - 2\right)^{1/2} - \sqrt{2}\left(\frac{1}{4}\omega^2 L_1 C_1 + \frac{4}{\omega^2 L_1 C_1} - 2\right)^{1/4} + 1\right]^2}\right]$

Examination of this equation shows $\alpha = 0$ when $\omega^2 = \frac{4}{L_1C_1}$, the cutoff frequency of the arms. Also, $\alpha \to 0$ when $\omega \to 0$ or $\omega \to \infty$.

Appendix D proves the maximum attenuation given by (36) is when $y_0 = 1$ and (37) proves the attenuation is then 7.64 db. The frequencies of maximum attenuation are

(111) $\frac{1}{4}\omega^2 L_1C_1 + \frac{4}{\omega^2 L_1C_1} - 2 = 1$

the solution of which is $\omega = \frac{3.24}{\sqrt{L_1C_1}}$ or $\frac{1.23}{\sqrt{L_1C_1}}$

Fig. 34 is a sketch of the attenuation for this filter.

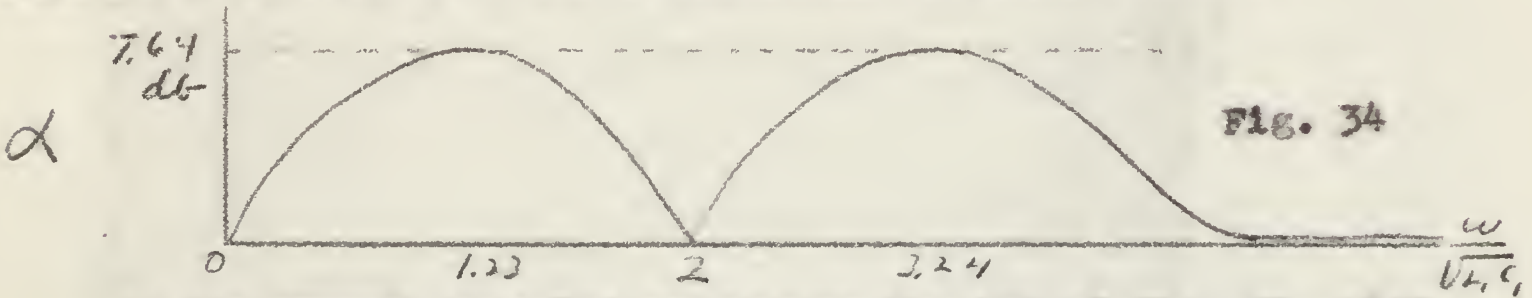


Fig. 34

As the second case let us assume the tee of fig. 35, with series arms high-pass pi terminated and shunt arm low-pass tee terminated.

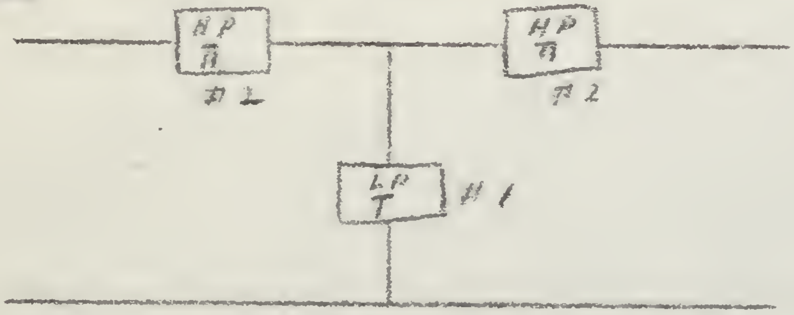


Fig. 35

(112) $Z_T = K\sqrt{1+P_1}$

(113) $Z_\pi = \frac{K}{\sqrt{1+P_2}}$

(114) $\rho = \frac{2Z_\pi}{4Z_T} = \frac{\frac{K}{\sqrt{1+P_2}}}{2K\sqrt{1+P_1}} = \frac{1}{2\sqrt{1+P_1}\sqrt{1+P_2}} = \frac{1}{2\sqrt{1-\frac{1}{4}\omega^2L_1C_1}\sqrt{1-\frac{1}{4\omega^2L_2C_2}}}$

Assume the same cutoff frequency for both arms.

(107) $L_2C_2 = \frac{L_1C_1}{16}$ Therefore

(115) $\rho = \frac{1}{2\sqrt{1-\frac{1}{4}\omega^2L_1C_1}\sqrt{1-\frac{4}{\omega^2L_1C_1}}}$

Below $\omega_{c1} = \omega_{c2}$

(116) $\rho = -j \cdot (\text{real}) = \text{negative imaginary}$

Above $\omega_{c1} = \omega_{c2}$

(117) $\rho = -j \cdot (\text{real}) = \text{negative imaginary}$

Hence, ρ is always a negative imaginary, and

(99) $\alpha = \cosh^{-1}\left[\sqrt{\rho^2+1} + \rho\right]$ where

(118) $$\rho = \frac{1}{2\sqrt{\left|1-\frac{1}{4}\omega^2 L_1 C_1\right|}\sqrt{\left|1-\frac{4}{\omega^2 L_1 C_1}\right|}}$$

Fig. 36 is a sketch of the attenuation. There is no pass band. For this case

(119) $$Z = \sqrt{2Z_\pi Z_T + \frac{(2Z_\pi)^2}{4}} = K\sqrt{2\frac{\sqrt{1-\frac{1}{4}\omega^2 L_1 C_1}}{\sqrt{1-\frac{4}{\omega^2 L_1 C_1}}} + \frac{1}{1-\frac{4}{\omega^2 L_1 C_1}}}$$

The impedance is a complex number for all frequencies except Z=K at $\omega = 0$. It has an infinity at $\omega = \frac{2}{\sqrt{L_1 C_1}}$.

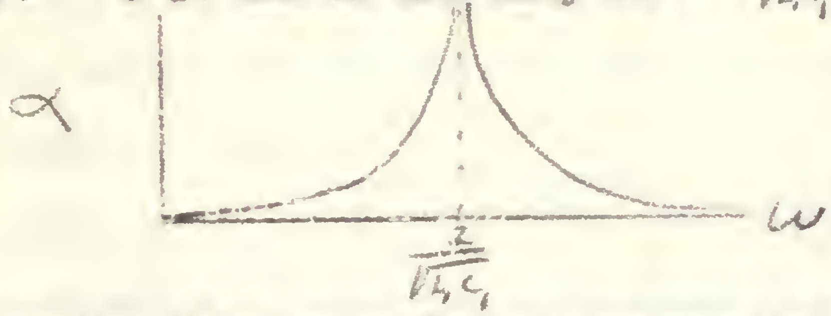


Fig. 36

For the third case let us assume the tee of fig. 37, with series arms high-pass pi terminated and the shunt arm low-pass pi terminated.

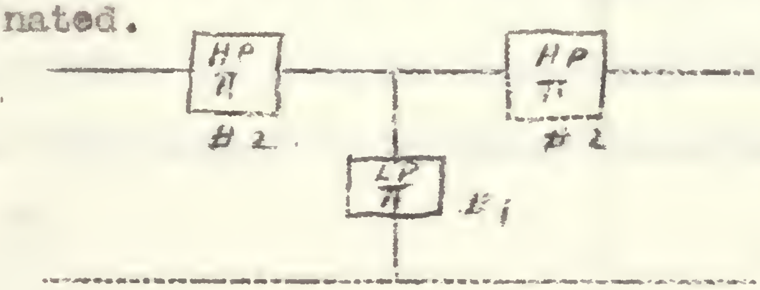


Fig. 37

(120) $$Z_2 = \frac{K}{\sqrt{1+\rho_2}} = \frac{K}{\sqrt{1-\frac{1}{4\omega^2 L_2 C_2}}}$$

(121) $$Z_1 = \frac{K}{\sqrt{1+\rho_1}} = \frac{K}{\sqrt{1-\frac{1}{4}\omega^2 L_1 C_1}}$$

Assuming $\omega_{c1} = \omega_{c2}$

(107) $L_2 C_2 = \frac{L_1 C_1}{16}$ then

(122) $Z_2 = \frac{K}{\sqrt{1-\frac{4}{\omega^2 L_1 C_1}}}$

(123) $\rho = \frac{2Z_2}{4Z_1} = \frac{\frac{K}{\sqrt{1-\frac{4}{\omega^2 L_1 C_1}}}}{\frac{2K}{\sqrt{1-\frac{1}{4}\omega^2 L_1 C_1}}} = \frac{1}{2}\frac{\sqrt{1-\frac{1}{4}\omega^2 L_1 C_1}}{\sqrt{1-\frac{4}{\omega^2 L_1 C_1}}}$

ρ is a negative imaginary below ω_c and a positive imaginary above ω_c. (98) reduces to

(99) $\alpha = \cosh^{-1}\left[\sqrt{\rho^2+1}+\rho\right]$

for all frequencies because the cosine of a negative angle is equal to the cosine of the positive angle. Hence

(124) $\rho = \frac{1}{2}\sqrt{\frac{1-\frac{1}{4}\omega^2 L_1 C_1}{1-\frac{4}{\omega^2 L_1 C_1}}}$

This expression appears indeterminate at $\omega = \frac{2}{\sqrt{L_1 C_1}}$, but can be evaluated by differentiating numerator and denominator of the quantity under the radical to give, at $\omega = \frac{2}{\sqrt{L_1 C_1}}$,

(125) $\rho = \frac{1}{2}$

(126) $Z = \sqrt{2Z_1 Z_2 + \frac{(2Z_2)^2}{4}} = K\sqrt{\frac{2}{\sqrt{1-\frac{1}{4}\omega^2 L_1 C_1}\sqrt{1-\frac{4}{\omega^2 L_1 C_1}}} + \frac{1}{1-\frac{1}{4}\omega^2 L_1 C_1}}$

The impedance is a complex number for all frequencies except $Z=K$ at $\omega=0$. It has an infinity at $\omega = \frac{2}{\sqrt{L_1 C_1}}$.

Fig. 38 is a sketch of the attenuation for this arrangement. The attenuation is zero at zero frequency and increases to infinite attenuation at infinite frequency.

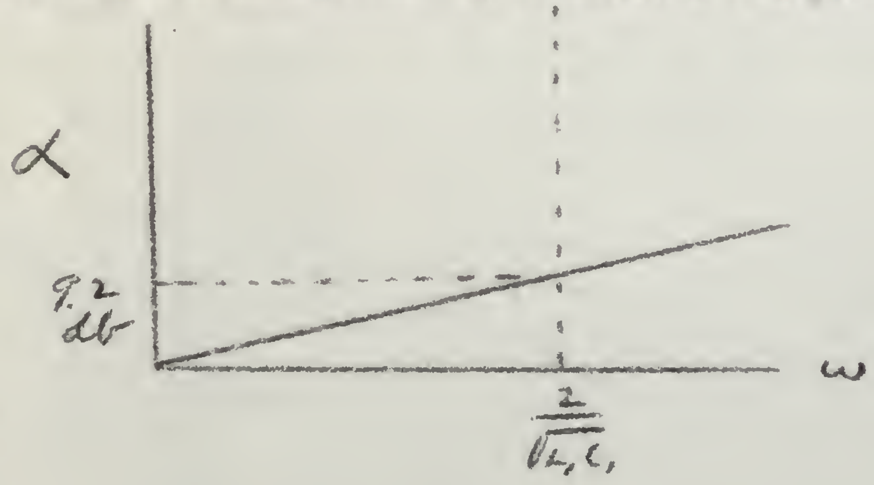


Fig. 38

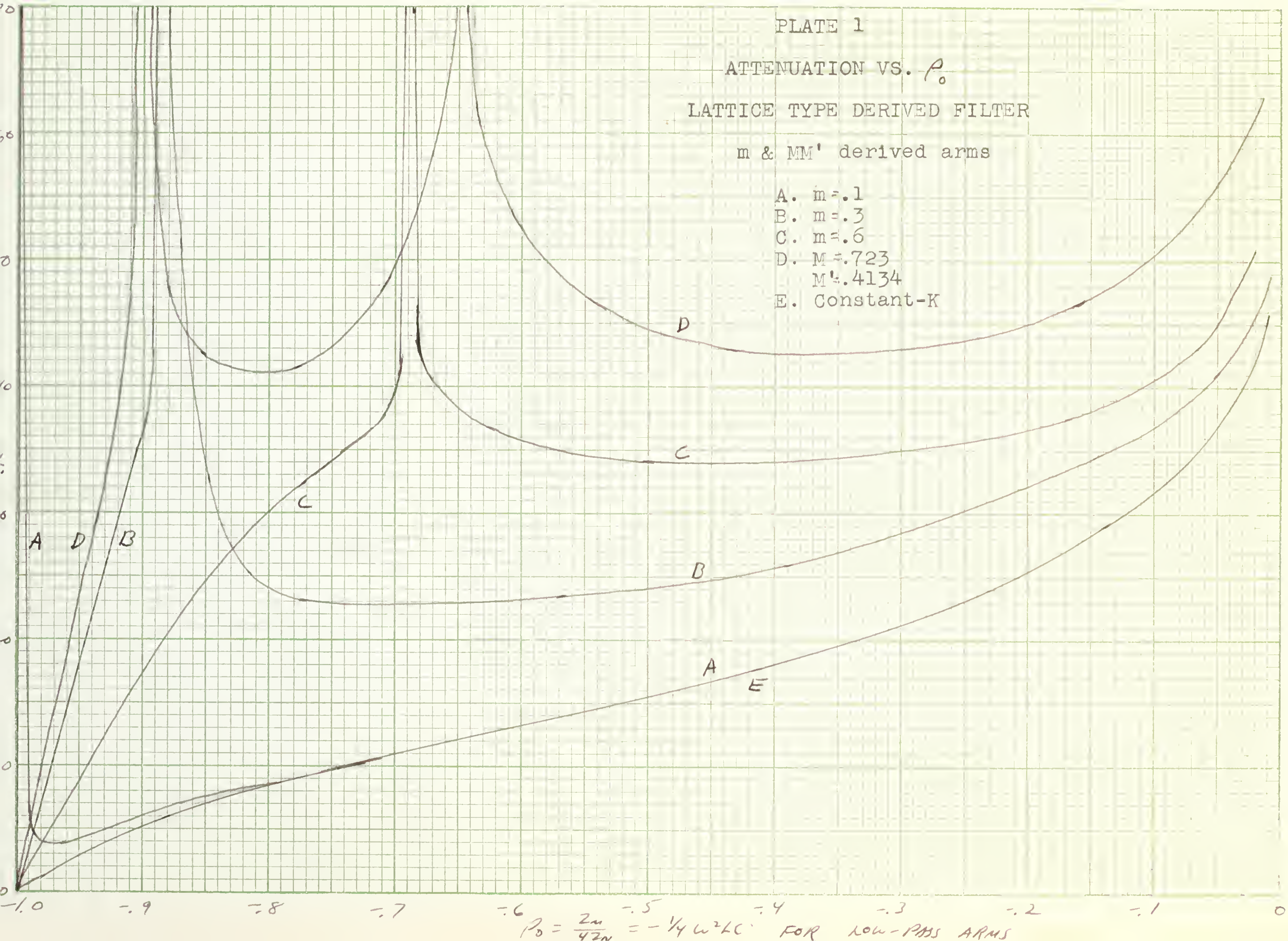


PLATE 1

ATTENUATION VS. ρ_o

LATTICE TYPE DERIVED FILTER

m & MM' derived arms

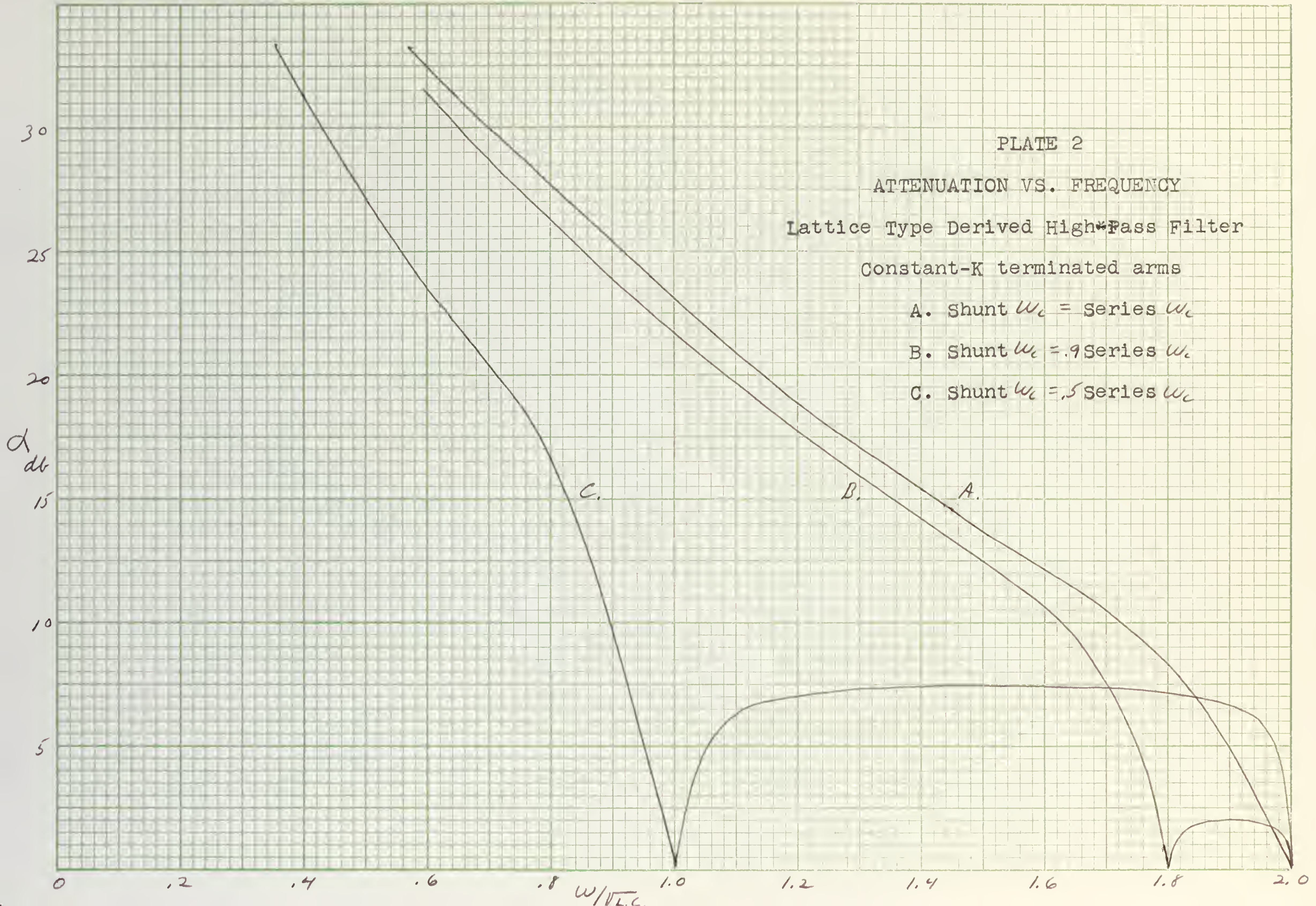


PLATE 2

ATTENUATION VS. FREQUENCY

Lattice Type Derived High-Pass Filter

Constant-K terminated arms

A. Shunt ω_c = Series ω_c

B. Shunt ω_c =.9 Series ω_c

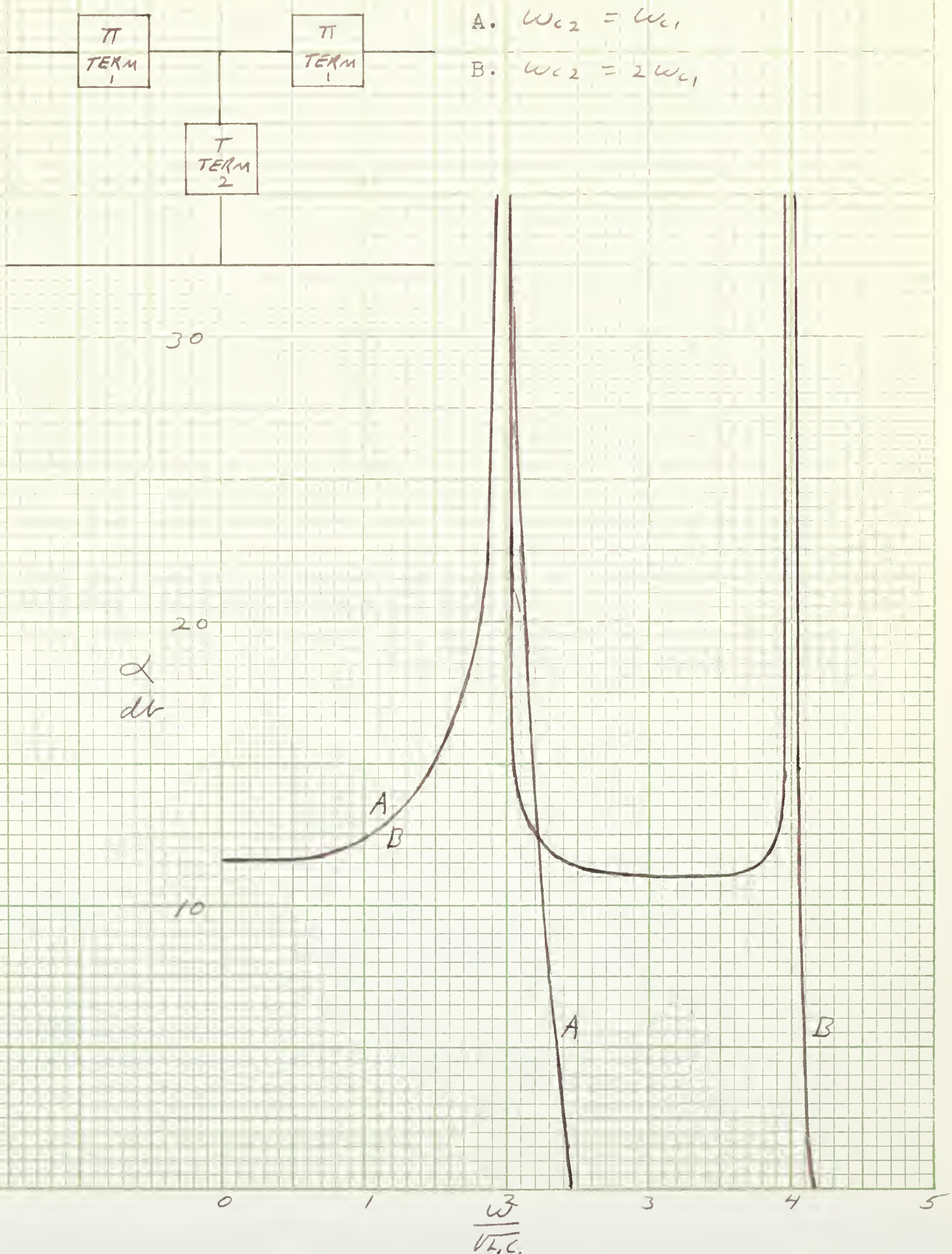
C. Shunt ω_c =.5 Series ω_c

PLATE 3

ATTENUATION VS. FREQUENCY

Tee Type Derived High-Pass Filter

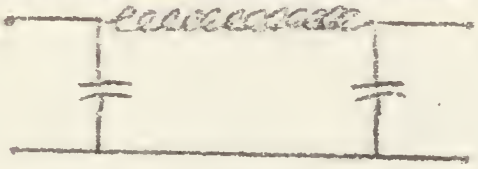
Constant-K terminated arms



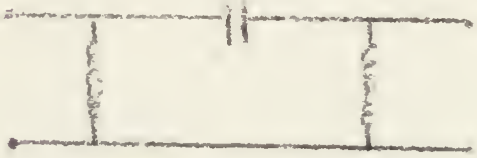
APPENDIX A

TYPES OF ARMS

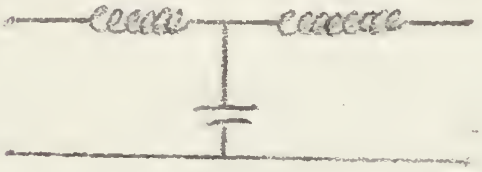
The left terminals join the filter. The right terminals are connected to a terminating section which is connected to a resistor. See *6 for numerical values.



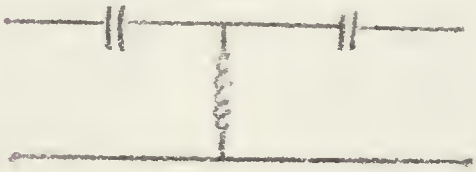
Low-pass π



High-pass π

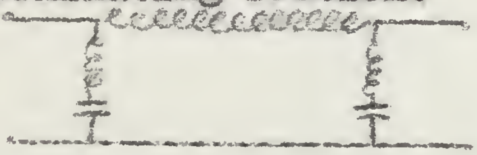


Low-pass T

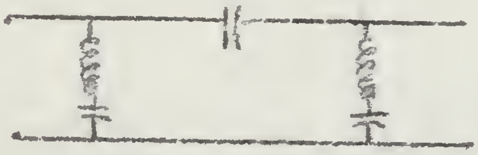


High-pass T

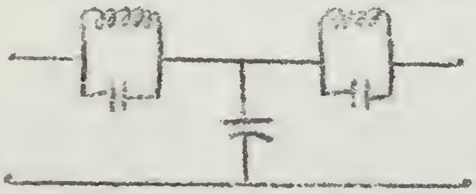
The remaining sections may be connected to the resistor without any other terminating section.



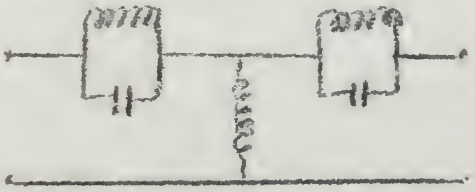
Low-pass π'
(m-derived)



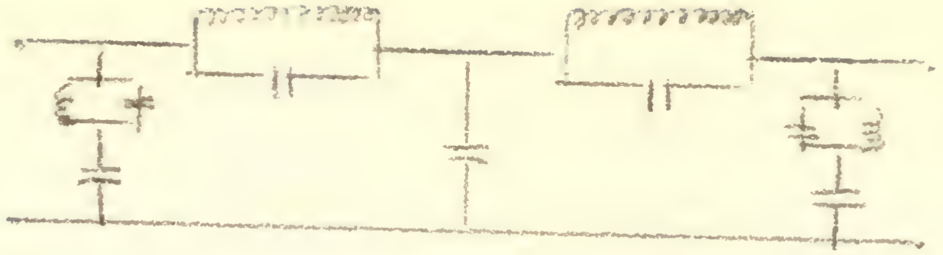
High-pass π'



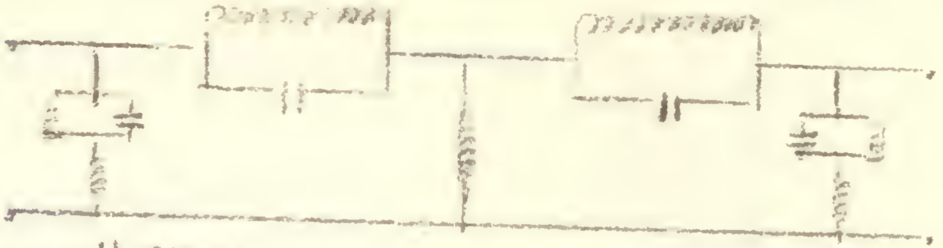
Low-pass T'



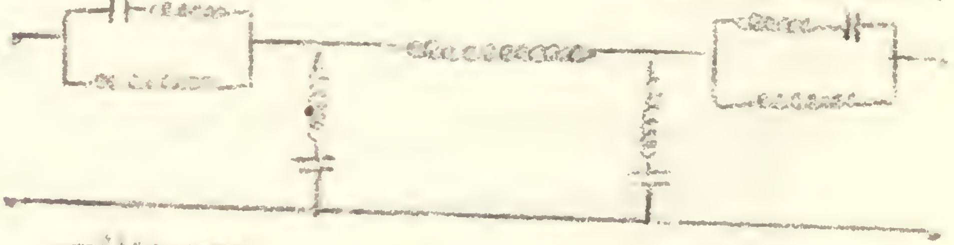
High-pass T'



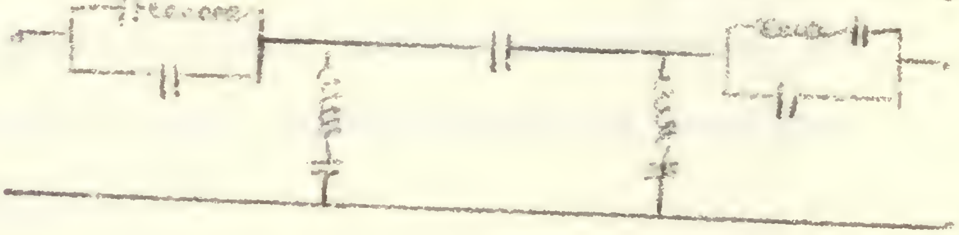
Low-pass π''
(MM'derived)



High-pass π''



Low-pass T''



High-pass T''

APPENDIX B

SYMBOLS

Symbol	Meaning	Page first used
α	Attenuation, nepers, unless specified db.	5
β	Phase	13
γ	Transfer function $\gamma = \alpha + j\beta$	13
ρ	One-fourth ratio of total series impedance of filter section to total shunt impedance of filter section.	15
$\boldsymbol{\rho}$	A complex ρ . Then ρ is the modulus of $\boldsymbol{\rho}$.	26
ρ_0	One-fourth ratio of total series impedance of arm section to total shunt impedance of arm section.	4
φ	The phase angle of $\boldsymbol{\rho}$	26
ω	Angular frequency	6
ω_c	Cutoff angular frequency	6
k	A constant	11
K	Square root of product Z_aZ_b	4
y	$\sqrt{\frac{Z_b}{Z_a}}$ or $\sqrt{\frac{Z_a}{Z_b}}$ of a lattice section	12
Z	Iterative impedance of derived filter section	16
Z_a	Total series impedance of arm section	5
Z_b	Total shunt impedance of arm section	5
Z_L	Iterative impedance of a lattice section	6
Z_m	Total series impedance of any filter section	4
Z_n	Total shunt impedance of any filter section	4
Z_T	Iterative impedance of an arm, with arm arranged as tee	4
Z_π	Iterative impedance of an arm, with arm arranged as pi	5
Z'_T	Mid-series image impedance of shunt-derived m-type	8
Z'_π	Mid-shunt image impedance of series-derived m-type	8

APPENDIX C

Proof that attenuation is zero when shunt and series arms of lattice are opposite reactances.

For any lattice[*2,p379]

(a) $\gamma = \alpha + j\beta = \ln\left(\frac{\sqrt{\frac{Z_b}{Z_a}}+1}{\sqrt{\frac{Z_b}{Z_a}}-1}\right)$

But we assume that Z_b and Z_a are opposite reactances, or that

(b) $\frac{Z_b}{Z_a} = -\text{Real} = -y_0^2$

(c) $\therefore \sqrt{\frac{Z_b}{Z_a}} = jy_0$

(d) $\therefore \gamma = \ln\left(\frac{jy_0+1}{jy_0-1}\right) = \ln\left[\left(\frac{1+jy_0}{-1+jy_0}\right)\left(\frac{-1-jy_0}{-1-jy_0}\right)\right]$

$$= \ln\left[\frac{-1+y_0^2+j(-2y_0)}{1+y_0^2}\right]$$

If $\alpha + j\beta = \ln(A+jB)$, then[*5]

(e) $\alpha = \ln\sqrt{A^2+B^2} = \frac{1}{2}\ln(A^2+B^2)$

(f) $\therefore \alpha = \frac{1}{2}\ln\left[\frac{-1+y_0^2 + (-2y_0)^2}{(1+y_0^2)^2}\right]$

$$= \frac{1}{2}\ln\left[\frac{1-2y_0^2+y_0^4+4y_0^2}{(1+y_0^2)^2}\right]$$

$$= \frac{1}{2}\ln\left[\frac{(1+y_0^2)^2}{(1+y_0^2)^2}\right]$$

$$= \frac{1}{2}\ln 1$$

$$= 0$$

APPENDIX D

Finding maximum value of $\dfrac{y_0^4+1}{(y_0^2-\sqrt{2}\,y_0+1)^2}$

Differentiating and setting equal to zero:

$$(y_0^4+1)(2)(y_0^2-\sqrt{2}\,y_0+1)(2y_0-\sqrt{2}) = (y_0^2-\sqrt{2}\,y_0+1)^2(4y_0^3)$$

$$2(y_0^4+1)(2y_0-\sqrt{2}) = 4y_0^3(y_0^2-\sqrt{2}\,y_0+1)$$

$$2(2y_0^5-\sqrt{2}\,y_0^4+2y_0-\sqrt{2}) = 4y_0^5-4\sqrt{2}\,y_0^4+4y_0^3$$

$$-2\sqrt{2}\,y_0^4+4y_0-2\sqrt{2} = -4\sqrt{2}\,y_0^4+4y_0^3$$

$$2\sqrt{2}\,y_0^4-4y_0^3+4y_0-2\sqrt{2}=0$$

$$\sqrt{2}\,y_0^4-2y_0^3+2y_0-\sqrt{2}=0$$

$$y_0^4-\sqrt{2}\,y_0^3+2y_0-1=0$$

The real roots of this equation are $y_0=+1$ and $y_0=-1$.

Proof: $\dfrac{y_0^4-\sqrt{2}\,y_0^3+2y_0-1}{(y_0+1)(y_0-1)} = y_0^2-\sqrt{2}\,y_0+1$

We are concerned only with the value +1, however, because y_0 was assumed a real positive quantity when defined. The maximum value of our expression is therefore:

$$\frac{1+1}{(1-\sqrt{2}+1)^2} = \frac{2}{(2-\sqrt{2})^2}$$

VI BIBLIOGRAPHY

(1) Bode, H.W. "Network Analysis and Feedback Amplifier Design", D. Van Nostrand, 1945.

(2) Guillemin, E.A. "Communication Networks", Vol. II, Wiley, 1935.

(3) McIlwain, K. & Brainerd, J.G. "High Frequency Alternating Currents", Wiley, 1939

(4) Shea, T.E. "Transmission Networks and Wave Filters", D. Van Nostrand, 1939.

(5) Sokolnikoff, I.S. & E.S. "Higher Mathematics for Engineers and Physicists", McGraw-Hill, 1941.

(6) Terman, F.E. "Radio Engineers Handbook",McGraw-Hill,1943.

(7) Zobel, O.J. "Extensions to the Theory and Design of Electric Wave Filters", BELL SYSTEM TECHNICAL JOURNAL, Vol. 10, April, 1931.

thes082
Electric wave filters having terminated
3 2768 001 00081 3
DUDLEY KNOX LIBRARY